Some Contagious Distributions Based on Continuous Distributions

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1 Introduction

"Contagious distribution" is named for the probability distribution of the sum

(1) $$S_N = X_0 + X_1 + X_2 + \ldots + X_N$$

where, X_n for $n = 0, 1, 2, \ldots$ is the random variable and the number N is also a random variable. A well known example of the contagious distribution is following; Let $X_1, X_2, \ldots$ be random variables with the Bernoulli distribution with $p = Pr(X_i = 1)$ for $i = 1, 2, \ldots$ and let N be the variable following Poisson distribution with the mean λ,then the sum $X_1 + X_2 + \ldots + X_N$ distributes as Poisson distribution with mean $p\lambda$.
This type of the contagious distribution, i.e., the number variable N is distributed as Poisson distribution is referred as "Poisson-stopped-sum -distribution". The significance of this type of distribution resides in the infinite divisibility by discrete distributions. There are already works on the Poisson-stopped-sum-distribution for the case that the number variable N as well as X-variables are discrete types in [1],[2] and [3].
Kanazawa developed the Poisson-stopped-sum-distribution for the case the X-variables are continuous as governed by a Normal distribution, and analyzed an archeological human bone data [4].
In the present paper to make the application of the contagious distribution feasible, we provide expressions for combinations of two distributions, that of the number variable N other than Poisson distribution and that of continuous X-variables of other than Normal distribution.

2 Method and Basic Distributions

We consider here the four contagious distributions that are obtained by two types of stopped-sum-distributions and two types of continuous distributions, which are interested in practical viewpoints. Those are Poisson distribution and Binomial distribution for the random variable N, i.e.

(Po) Poisson-stopped-sum-distribution
(Bi) Binomial-stopped-sum-distribution (we call it by this name in this paper)

and the X-variables follow to

(N) Normal distribution
(G) Gamma distribution.

Notations
We deal with the following probability distributions. For the stopped-sum-distributions (Po) and (Bi) are as follows.
Po(λ) ($\lambda > 0$) is the Poisson distribution on $n = 0, 1, 2, \ldots$ with the density

$$p(n) = \frac{\lambda^n}{n!} \exp(-\lambda)$$

and
B(n, p) ($n \geq 1, 0 < p < 1$) is the Binomial distribution on $k = 0, 1, \ldots, n$ with the density

$$p(k) = \binom{n}{k} p^k q^{n-k} \qquad (q = 1 - p) \ .$$

For the based distributions are followings.
N(μ, σ^2) is the Normal distribution on $x \in \mathbf{R}$ with the density

$$p(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp(-\frac{(x-\mu)^2}{2\sigma^2})$$

and
G(α, β) ($\alpha > 0, \beta > 0$) is the Gamma distribution on $x \in [0, \infty)$ with the density

$$p(x) = \frac{\alpha^\beta}{\Gamma(\beta)} x^{\beta-1} \exp(-\alpha x) \ .$$

Those four contagious distributions are represented by following three terms.

(1) the density function
(2) the moment generating function
(3) moments.

It happens that we can not observe the X_0-variable itself in many practical works. For this case we propose to treat the X_0-variable by using Dirac's delta function $\delta_0(x)$. Those are given in Case $S_{1,N}$. The density function S_N itself of (1) is given in Case $S_{0,N}$.
Random variable N follows a Poisson-stopped-sum-distribution in the following 2.1 and 2.2 and a Binomial-stopped-sum-distribution in the following 2.3 and 2.4.

2.1 The contagious distribution based on $\mathbf{N}(\mu, \sigma^2)$ for the stopped-sum (Po)

2.1.1 Case $S_{1,N}$

The density is given by

$$(2) \quad f_1(x) = \exp(-\lambda)\delta_0(x) + \sum_{n=1}^{\infty} \frac{\lambda^n}{n!} \exp(-\lambda) \frac{1}{\sqrt{2\pi n\sigma^2}} \exp(-\frac{(x-n\mu)^2}{2n\sigma^2})$$

with Dirac's delta function $\delta_0(x)$ at $x = 0$. The moment generating function is

$$(3) \qquad M_1(\theta) = \exp(\lambda \exp(\mu\theta + \frac{\sigma^2}{2}\theta^2) - \lambda) \ .$$

We set

$$(4) \qquad m_n = \int_{-\infty}^{\infty} x^n f_1(x) dx \quad n \geq 1$$

$$(5) \qquad v_n = \int_{-\infty}^{\infty} (x - m_1)^n f_1(x) dx \quad n \geq 1 \ .$$

From differentiating (3), we obtain

$$(6) \qquad m_1 = \lambda\mu$$

$$(7) \qquad m_2 = (\lambda\mu)^2 + \lambda\mu^2 + \lambda\sigma^2$$

$$(8) \qquad m_3 = (\lambda\mu)^3 + 3\lambda\mu(\lambda\mu^2 + \lambda\sigma^2) + 3\lambda\mu\sigma^2 + \lambda\mu^3 \ .$$

Since

$$v_2 = m_2 - m_1^2$$

$$v_3 = m_3 - 3m_1m_2 + 2m_1^3$$

we have

(9) $$v_2 = \lambda\mu^2 + \lambda\sigma^2$$

(10) $$v_3 = \lambda\mu(\mu^2 + 3\sigma^2) \ .$$

By (6),(9) and (10), we have

$$v_2\mu = m_1(\mu^2 + \sigma^2)$$

$$v_3 = m_1(\mu^2 + 3\sigma^2) \ .$$

Hence μ is the solution of

$$2m_1\mu^2 - 3v_2\mu + v_3 = 0 \ .$$

Then μ is determined by m_1, m_2 and m_3. Hence ,by (6) and (9) ,λ and σ^2 are determined by m_1, m_2 and m_3.
If $\mu = 0$ then $m_1 = m_3 = 0$. Hence λ and σ^2 are not determined. In this case we consider m_2 and m_4,

$$m_2 = \lambda\sigma^2$$
$$m_4 = 3\lambda(1 + \lambda)\sigma^4$$

and we get

$$\lambda = \frac{3m_2^2}{m_4 - 3m_2^2}$$
$$\sigma^2 = \frac{m_4 - 3m_2^2}{3m_2} \ .$$

Hence λ and σ^2 are determined by m_2 and m_4 .

2.1.2 Case $S_{0,N}$

Since the density is given by

(11) $$f_2(x) = \sum_{n=0}^{\infty} \frac{\lambda^n}{n!} \exp(-\lambda) \frac{1}{\sqrt{2\pi(n+1)\sigma^2}} \exp(-\frac{(x-(n+1)\mu)^2}{2(n+1)\sigma^2}) \ ,$$

the moment generating function is

$$(12) \qquad M_2(\theta) = \exp(\mu\theta + \frac{\sigma^2}{2}\theta^2)M_1(\theta) \ .$$

We set

$$(13) \qquad \mu_n = \int_{-\infty}^{\infty} x^n f_2(x)dx \qquad n \geq 1$$

$$(14) \qquad \nu_n = \int_{-\infty}^{\infty} (x-\mu_1)^n f_2(x)dx \qquad n \geq 1 \ .$$

Expanding the both sides of (12) , we get

$$(15) \qquad 1 + \sum_{n=1}^{\infty} \frac{\mu_n}{n!}\theta^n = \exp(\mu\theta + \frac{\sigma^2}{2}\theta^2)(1 + \sum_{n=1}^{\infty} \frac{m_n}{n!}\theta^n) \ ,$$

where $m_n, n \geq 1$ are determined by (4).
Then we have

$$(16) \qquad \mu_1 = m_1 + \mu$$

$$(17) \qquad \mu_2 = m_2 + 2\mu m_1 + \mu^2 + \sigma^2$$

$$(18) \qquad \mu_3 = m_3 + 3\mu m_2 + 3(\mu^2 + \sigma^2)m_1 + \mu^3 + 3\mu\sigma^2 \ .$$

Since

$$\nu_2 = \mu_2 - \mu_1^2$$

$$\nu_3 = \mu_3 - 3\mu_1\mu_2 + 2\mu_1^3$$

we obtain

$$(19) \qquad \begin{aligned} \nu_2 &= v_2 + \sigma^2 \\ &= \lambda\mu^2 + \lambda\sigma^2 + \sigma^2 \quad \text{by (9)} \end{aligned}$$

and

$$(20) \qquad \begin{aligned} \nu_3 &= v_3 \\ &= \lambda\mu(\mu^2 + 3\sigma^2) \quad \text{by (10)} \ . \end{aligned}$$

By (6) , (16) and (19) we have

$$\lambda = \frac{\mu_1}{\mu} - 1$$

$$\sigma^2 = \frac{\mu}{\mu_1}(\nu_2 - \mu_1\mu + \mu^2)$$

and from (20) we obtain

$$\nu_3\mu_1 = \mu(\mu_1 - \mu)(3\nu_2 - 2\mu_1\mu + 3\mu^2) \ . \tag{21}$$

Since μ is the solution of (21),μ is determined by μ_1, μ_2 and μ_3.Then λ and σ^2 are determined by μ_1, μ_2 and μ_3.
If $\mu = 0$ then $\mu_1 = \mu_3 = 0$. Hence λ and σ^2 are not determined. In this case we consider μ_2 and μ_4,

$$\mu_2 = (1+\lambda)\sigma^2$$
$$\mu_4 = 3(\lambda^2 + 3\lambda + 1)\sigma^4$$

and λ is the solution of

$$(\mu_4 - 3\mu_2^2)\lambda^2 + (2\mu_4 - 9\mu_2^2)\lambda + (\mu_4 - 3\mu_2^2) = 0 \ .$$

We obtain λ and σ^2 by μ_2 and μ_4 .

2.2 The contagious distribution based on G(α, β) for stopped-sum (Po)

2.2.1 Case $S_{1,N}$

Since the density is given by

$$f_1(x) = \exp(-\lambda)\delta_0(x) + \sum_{n=1}^{\infty} \frac{\lambda^n}{n!} \exp(-\lambda) \frac{\alpha^{n\beta}}{\Gamma(n\beta)} x^{n\beta-1} \exp(-\alpha x) \ , \tag{22}$$

the moment generating function is

$$\begin{aligned} M_1(\theta) &= \exp(\lambda(1 - \frac{\theta}{\alpha})^{-\beta} - \lambda) \\ &= \exp(\lambda \sum_{n=1}^{\infty} \binom{\beta+n-1}{n} (\frac{\theta}{\alpha})^n) \ . \end{aligned} \tag{23}$$

We set

$$m_n = \int_0^{\infty} x^n f_1(x)dx \quad n \geq 1 \tag{24}$$

(25) $$v_n = \int_0^\infty (x - m_1)^n f_1(x) dx \quad n \geq 1 \ .$$

From differentiating (23) ,we obtain

(26) $$m_1 = \frac{\lambda\beta}{\alpha}$$

(27) $$\begin{aligned} m_2 &= (\frac{\lambda\beta}{\alpha})^2 + \frac{\lambda\beta(\beta+1)}{\alpha^2} \\ &= m_1^2 + m_1 \frac{\beta+1}{\alpha} \end{aligned}$$

(28) $$\begin{aligned} m_3 &= (\frac{\lambda\beta}{\alpha})^3 + 3(\frac{\lambda\beta}{\alpha})(\frac{\lambda\beta(\beta+1)}{\alpha^2}) + \frac{\lambda\beta(\beta+1)(\beta+2)}{\alpha^3} \\ &= m_1^3 + 3m_1^2 \frac{\beta+1}{\alpha} + m_1 \frac{(\beta+1)(\beta+2)}{\alpha^2} \ . \end{aligned}$$

Since

$$v_2 = m_2 - m_1^2$$

$$v_3 = m_3 - 3m_1 m_2 + 2m_1^3$$

we obtain

(29) $$v_2 = m_1 \frac{\beta+1}{\alpha}$$

(30) $$v_3 = m_1 \frac{(\beta+1)(\beta+2)}{\alpha^2} \ .$$

By (29) we have

$$\frac{\beta+1}{\alpha} = \frac{v_2}{m_1} \ .$$

Then from (30) we have

$$v_3 = v_2 (\frac{v_2}{m_1} + \frac{1}{\alpha})$$

and

$$\alpha = \frac{m_1 v_2}{m_1 v_3 - v_2^2} \ .$$

Hence,by (26) and (29), α,β and λ are determined by m_1, m_2 and m_3.

2.2.2 Case $S_{0,N}$

Since the density is given by

$$(31) \qquad f_2(x) = \sum_{n=0}^{\infty} \frac{\lambda^n}{n!} \exp(-\lambda) \frac{\alpha^{(n+1)\beta}}{\Gamma((n+1)\beta)} x^{(n+1)\beta-1} \exp(-\alpha x) \ ,$$

the moment generating function is

$$(32) \qquad M_2(\theta) = (1 - \frac{\theta}{\alpha})^{-\beta} M_1(\theta) \ .$$

We set

$$(33) \qquad \mu_n = \int_0^{\infty} x^n f_2(x) dx \quad n \geq 1$$

$$(34) \qquad \nu_n = \int_0^{\infty} (x - \mu_1)^n f_2(x) dx \quad n \geq 1 \ .$$

Expanding the both sides of (32) ,we get

$$(35) \quad 1 + \sum_{n=1}^{\infty} \frac{\mu_n}{n!} \theta^n = (1 + \sum_{n=1}^{\infty} \begin{pmatrix} \beta + n - 1 \\ n \end{pmatrix} (\frac{\theta}{\alpha})^n)(1 + \sum_{n=1}^{\infty} \frac{m_n}{n!} \theta^n) \ ,$$

where $m_n, n \geq 1$ are determined by (24). Then we have

$$(36) \qquad \mu_1 = m_1 + \frac{\beta}{\alpha}$$

$$(37) \qquad \mu_2 = m_2 + 2m_1 \frac{\beta}{\alpha} + \frac{\beta(\beta+1)}{\alpha^2}$$

$$(38) \qquad \mu_3 = m_3 + 3m_2 \frac{\beta}{\alpha} + 3m_1 \frac{\beta(\beta+1)}{\alpha^2} + \frac{\beta(\beta+1)(\beta+2)}{\alpha^3} \ .$$

Since

$$\nu_2 = \mu_2 - \mu_1^2$$

$$\nu_3 = \mu_3 - 3\mu_1\mu_2 + 2\mu_1^3$$

we have

$$(39) \qquad \begin{aligned} \nu_2 &= v_2 + \frac{\beta}{\alpha^2} \\ &= \frac{m_1}{\alpha}(\beta+1) + \frac{\beta}{\alpha^2} \quad \text{by (29)} \\ &= \mu_1 \frac{\beta+1}{\alpha} - (\frac{\beta}{\alpha})^2 \quad \text{by (36)} \end{aligned}$$

and

$$(40) \qquad \begin{aligned} \nu_3 &= v_3 + \frac{2\beta}{\alpha^3} \\ &= (\mu_1 - \frac{\beta}{\alpha})\frac{(\beta+1)(\beta+2)}{\alpha^2} + \frac{2\beta}{\alpha^3} \end{aligned}$$

by (30) and (36). Then we have

$$(41) \qquad \beta^2 - \mu_1\alpha\beta + \nu_2\alpha^2 - \mu_1\alpha = 0$$

and

$$(42) \qquad \beta^3 + (3 - \alpha\mu_1)\beta^2 - 3\alpha\mu_1\beta + (\nu_3\alpha^3 - 2\alpha\mu_1) = 0 \; .$$

Since

$$\begin{aligned} \beta^3 &+ (3 - \alpha\mu_1)\beta^2 - 3\alpha\mu_1\beta + (\nu_3\alpha^3 - 2\alpha\mu_1) \\ &= (\beta + 3)(\beta^2 - \mu_1\alpha\beta + \nu_2\alpha^2 - \mu_1\alpha) \\ &\quad +(\alpha\mu_1 - \nu_2\alpha^2)\beta + \alpha\mu_1 + \nu_3\alpha^3 - 3\nu_2\alpha^2 \end{aligned}$$

we have

$$(43) \qquad \beta = \frac{-\mu_1 - \nu_3\alpha^2 + 3\nu_2\alpha}{\mu_1 - \nu_2\alpha} \; .$$

From (43) and (41), α satisfies

$$(44) \qquad (\mu_1 - \nu_2\alpha)(\mu_1\nu_3\alpha - \nu_2^2\alpha - \mu_1\nu_2)\alpha^2 + (\nu_3\alpha^2 - 3\nu_2\alpha + \mu_1)^2 = 0 \; .$$

Since α is the solution of (44) α is determined by μ_1, μ_2 and μ_3. Then, by (43),(26) and (36), β and λ are determined by μ_1,μ_2 and μ_3.

2.3 The contagious distribution based on N(μ, σ^2) for stopped-sum (Bi)

2.3.1 Case $S_{1,N}$

Since the density is given by

$$(45) \qquad f_1(x) = q^n\delta_0(x) + \sum_{k=1}^{n} \binom{n}{k} p^k q^{n-k} \frac{1}{\sqrt{2\pi k\sigma^2}} \exp(-\frac{(x-k\mu)^2}{2k\sigma^2}) \; ,$$

the moment generating function is

$$(46)\qquad \begin{aligned} M_1(\theta) &= (q + p\exp(\mu\theta + \frac{\sigma^2}{2}\theta^2))^n \\ &= (1 + \sum_{k=1}^{\infty} \frac{p}{k!}(\mu\theta + \frac{\sigma^2}{2}\theta^2)^k)^n \\ &= 1 + \sum_{j=1}^{n} \binom{n}{j} (\sum_{k=1}^{\infty} \frac{p}{k!}(\mu\theta + \frac{\sigma^2}{2}\theta^2)^k)^j \ . \end{aligned}$$

We set

$$(47)\qquad m_n = \int_{-\infty}^{\infty} x^n f_1(x)dx \qquad n \geq 1$$

$$(48)\qquad v_n = \int_{-\infty}^{\infty} (x - m_1)^n f_1(x)dx \qquad n \geq 1 \ .$$

From differentiating (46),we obtain

$$(49)\qquad m_1 = np\mu$$

$$(50)\qquad \begin{aligned} m_2 &= n(n-1)(p\mu)^2 + np(\mu^2 + \sigma^2) \\ &= \frac{n-1}{n} m_1^2 + np(\mu^2 + \sigma^2) \end{aligned}$$

$$(51)\qquad \begin{aligned} m_3 &= n(n-1)(n-2)(p\mu)^3 \\ &\quad + 3n(n-1)p\mu(p\mu^2 + p\sigma^2) + np\mu(\mu^2 + 3\sigma^2) \\ &= \frac{(n-1)(n-2)}{n^2} m_1^3 \\ &\quad + 3m_1(n-1)p(\mu^2 + \sigma^2) + m_1(\mu^2 + 3\sigma^2) \ . \end{aligned}$$

Since

$$v_2 = m_2 - m_1^2$$

$$v_3 = m_3 - 3m_1 m_2 + 2m_1^3$$

we have

$$(52)\qquad v_2 = -\frac{1}{n} m_1^2 + np(\mu^2 + \sigma^2)$$

$$(53)\qquad v_3 = \frac{2}{n^2} m_1^3 - 3pm_1(\mu^2 + \sigma^2) + m_1(\mu^2 + 3\sigma^2) \ .$$

By (49) and (52) we have

$$\mu = \frac{m_1}{np}$$

$$\sigma^2 = \frac{npv_2 - qm_1^2}{n^2p^2} .$$

Then by (53) we have

$$(n^2v_3 + m_1^3 + 3nm_1v_2)p^2 - 3m_1(nv_2 + m_1^2)p + 2m_1^3 = 0 . \tag{54}$$

Since p is the solution of (54),p is determined by n, m_1, m_2 and m_3. Hence μ and σ^2 are determined by n, m_1, m_2 and m_3.
If $\mu = 0$ then $m_1 = m_3 = 0$. Hence p and σ^2 are not determined. In this case we consider m_2 and m_4,

$$m_2 = np\sigma^2$$

$$m_4 = 3np(1 + (n-1)p)\sigma^4$$

and we have

$$p = \frac{3m_2^2}{nm_4 - 3(n-1)m_2^2}$$

$$\sigma^2 = \frac{nm_4 - 3(n-1)m_2^2}{3nm_2} .$$

Hence p and σ^2 are determined by n,m_2 and m_4.

2.3.2 Case $S_{0,N}$

Since the density is given by

$$f_2(x) = \sum_{k=0}^{n} \binom{n}{k} p^k q^{n-k} \frac{1}{\sqrt{2\pi(k+1)\sigma^2}} \exp(-\frac{(x-(k+1)\mu)^2}{2(k+1)\sigma^2}) , \tag{55}$$

the moment generating function is

$$M_2(\theta) = \exp(\mu\theta + \frac{\sigma^2}{2}\theta^2)M_1(\theta) . \tag{56}$$

We set

$$\mu_n = \int_{-\infty}^{\infty} x^n f_2(x)dx \qquad n \geq 1 \tag{57}$$

(58) $$\nu_n = \int_{-\infty}^{\infty} (x-\mu_1)^n f_2(x)dx \quad n \geq 1 \ .$$

Expanding the both sides of (56),we obtain

(59) $$1 + \sum_{n=1}^{\infty} \frac{\mu_n}{n!}\theta^n = \exp(\mu\theta + \frac{\sigma^2}{2}\theta^2)(1 + \sum_{n=1}^{\infty} \frac{m_n}{n!}\theta^n) \ ,$$

where $m_n, n \geq 1$ are determined by (47).Then we have

(60) $$\begin{aligned} \mu_1 &= m_1 + \mu \\ &= (np+1)\mu \quad \text{by} \ (49) \end{aligned}$$

(61) $$\begin{aligned} \mu_2 &= m_2 + 2\mu m_1 + \mu^2 + \sigma^2 \\ &= (1+np)^2\mu^2 + (1+np)\sigma^2 + npq\mu^2 \end{aligned}$$

by (49) and (50),

(62) $$\begin{aligned} \mu_3 &= m_3 + 3\mu m_2 + 3(\mu^2+\sigma^2)m_1 + \mu^3 + 3\mu\sigma^2 \\ &= \mu^3((1+np)^3 + npq(3np+4-2p)) \\ &+ \ 3\mu\sigma^2((1+np)^2 + npq) \end{aligned}$$

by (49),(50) and (51). Since

$$\nu_2 = \mu_2 - \mu_1^2$$

$$\nu_3 = \mu_3 - 3\mu_1\mu_2 + 2\mu_1^3$$

we obtain

(63) $$\nu_2 = (1+np)\sigma^2 + npq\mu^2$$

and

(64) $$\nu_3 = \mu^3 npq(1-2p) + 3\mu\sigma^2 npq \ .$$

By (60) and (63) we have

(65) $$\mu = \frac{\mu_1}{np+1}$$

(66) $$\sigma^2 = \frac{\nu_2}{1+np} - \frac{npq\mu_1^2}{(1+np)^3}$$

and from (64) we obtain

(67) $$\nu_3(1+np)^4 = 3\mu_1\nu_2 npq(1+np)^2 + \mu_1^3 npq(1-2p-2np+np^2) \ .$$

Since p is the solution of (67), p is determined by n, μ_1, μ_2 and μ_3. Then μ and σ^2 are determined by n, μ_1, μ_2 and μ_3.
If $\mu = 0$ then $\mu_1 = \mu_3 = 0$. Hence p and σ^2 are not determined. In this case we consider μ_2 and μ_4,

$$\mu_2 = (1+np)\sigma^2$$
$$\mu_4 = 3(1+3np+n(n-1)p^2)\sigma^4$$

and p is the solution of

$$\mu_4(1+np)^2 = 3\mu_2^2(1+3np+n(n-1)p^2) \ .$$

We obtain p and σ^2 by n,μ_2 and μ_4.

2.4 The contagious distribution based on G(α, β) for stopped-sum (Bi)

2.4.1 Case $S_{1,N}$

Since the density is given by

$$(68) \qquad f_1(x) = q^n \delta_0(x) + \sum_{k=1}^{n} \binom{n}{k} p^k q^{n-k} \frac{\alpha^{k\beta}}{\Gamma(k\beta)} x^{k\beta-1} \exp(-\alpha x) \ ,$$

the moment generating function is

$$(69) \qquad \begin{aligned} M_1(\theta) &= (q + p(1-\frac{\theta}{\alpha})^{-\beta})^n \\ &= (1+\sum_{k=1}^{\infty} p \binom{\beta+k-1}{k} (\frac{\theta}{\alpha})^k)^n \\ &= 1+\sum_{j=1}^{n} \binom{n}{j} (\sum_{k=1}^{\infty} p \binom{\beta+k-1}{k} (\frac{\theta}{\alpha})^k)^j \ . \end{aligned}$$

We set

$$(70) \qquad m_n = \int_0^\infty x^n f_1(x)dx \qquad n \geq 1$$

$$(71) \qquad v_n = \int_0^\infty (x-m_1)^n f_1(x)dx \qquad n \geq 1 \ .$$

Differentiating (69), we obtain

$$(72) \qquad m_1 = np\frac{\beta}{\alpha}$$

(73)
$$m_2 = n(n-1)(\frac{p\beta}{\alpha})^2 + np\frac{\beta(\beta+1)}{\alpha^2} = \frac{n-1}{n}m_1^2 + \frac{\beta+1}{\alpha}m_1$$

(74)
$$m_3 = np\frac{\beta(\beta+1)(\beta+2)}{\alpha^3} + 3n(n-1)\frac{p^2\beta^2(\beta+1)}{\alpha^3} + n(n-1)(n-2)(\frac{p\beta}{\alpha})^3 = \frac{(\beta+1)(\beta+2)}{\alpha^2}m_1 + 3\frac{(n-1)}{n}\frac{(\beta+1)}{\alpha}m_1^2 + \frac{(n-1)(n-2)}{n^2}m_1^3 .$$

Since

(75)
$$v_2 = m_2 - m_1^2$$

(76)
$$v_3 = m_3 - 3m_1m_2 + 2m_1^3$$

we obtain

(77)
$$v_2 = -\frac{1}{n}m_1^2 + \frac{\beta+1}{\alpha}m_1$$

(78)
$$v_3 = \frac{2}{n^2}m_1^3 - \frac{3}{n}\frac{(\beta+1)}{\alpha}m_1^2 + \frac{(\beta+1)(\beta+2)}{\alpha^2}m_1 .$$

Then by (72) and (77) we obtain

$$\alpha = \frac{m_1np}{v_2np - qm_1^2}$$

$$\beta = \frac{m_1^2}{v_2np - qm_1^2} .$$

Then from (78) we have

$$p = \frac{m_1^2(nv_2 + m_1^2)}{2n^2v_2^2 + nm_1^2v_2 + m_1^4 - n^2m_1v_3} .$$

Hence p,α and β are determined by n, m_1, m_2 and m_3.

2.4.2 Case $S_{0,N}$

Since the density is given by

$$f_2(x) = \sum_{k=0}^{n} \binom{n}{k} p^k q^{n-k} \frac{\alpha^{(k+1)\beta}}{\Gamma((k+1)\beta)} x^{(k+1)\beta-1} \exp(-\alpha x) \ , \tag{79}$$

the moment generating function is

$$M_2(\theta) = (1-\frac{\theta}{\alpha})^{-\beta} M_1(\theta) \ . \tag{80}$$

We set

$$\mu_n = \int_0^\infty x^n f_2(x)dx \quad n \geq 1 \tag{81}$$

$$\nu_n = \int_0^\infty (x-\mu_1)^n f_2(x)dx \quad n \geq 1 \ . \tag{82}$$

Expanding both sides of (80),we get

$$1 + \sum_{n=1}^{\infty} \frac{\mu_n}{n!}\theta^n = (1 + \sum_{n=1}^{\infty} \binom{\beta+n-1}{n} (\frac{\theta}{\alpha})^n)(1 + \sum_{n=1}^{\infty} \frac{m_n}{n!}\theta^n) \ , \tag{83}$$

where $m_n, n \geq 1$ are determined by (70). Then we have

$$\mu_1 = m_1 + \frac{\beta}{\alpha} \tag{84}$$

$$\mu_2 = m_2 + 2m_1\frac{\beta}{\alpha} + \frac{\beta(\beta+1)}{\alpha^2} \tag{85}$$

$$\mu_3 = m_3 + 3m_2\frac{\beta}{\alpha} + 3m_1\frac{\beta(\beta+1)}{\alpha^2} + \frac{\beta(\beta+1)(\beta+2)}{\alpha^3} \ . \tag{86}$$

Since

$$\nu_2 = \mu_2 - \mu_1^2$$

$$\nu_3 = \mu_3 - 3\mu_1\mu_2 + 2\mu_1^3$$

we obtain

$$\begin{aligned} \nu_2 &= v_2 + \frac{\beta}{\alpha^2} \\ &= -np^2(\frac{\beta}{\alpha})^2 + np\frac{\beta(\beta+1)}{\alpha^2} + \frac{\beta}{\alpha^2} \end{aligned} \tag{87}$$

by (72) and (77),

$$(88) \quad \nu_3 = v_3 + \frac{2\beta}{\alpha^3}$$
$$= 2np^3(\frac{\beta}{\alpha})^3 - 3np^2\frac{\beta^2(\beta+1)}{\alpha^3} + np\frac{\beta(\beta+1)(\beta+2)}{\alpha^3} + \frac{2\beta}{\alpha^3}$$

by (72) and (78). By (72) and (84) we obtain

$$\frac{\beta}{\alpha} = \frac{\mu_1}{np+1} .$$

Then from (87)

$$(89) \qquad \alpha = \frac{\mu_1(np+1)^2}{\nu_2(np+1)^2 - npq\mu_1^2}$$

$$(90) \qquad \beta = \frac{\mu_1^2(np+1)}{\nu_2(np+1)^2 - npq\mu_1^2} .$$

From (88)

$$\nu_3\alpha^3 = -np(2p-1)q\beta^3 + 3npq\beta^2 + 2(np+1)\beta .$$

Hence p is the solution of

$$\nu_3\mu_1(np+1)^4 = -\mu_1^4 np(2p-1)q(np+1)$$
$$+3\mu_1^2 npq(\nu_2(np+1)^2 - \mu_1^2 npq) + 2(\nu_2(np+1)^2 - \mu_1^2 npq)^2 .$$

Then p is determined by n, μ_1, μ_2 and μ_3. Then ,by (89) and (90) ,α and β are determined by n, μ_1, μ_2 and μ_3.

3 Discussions

Mathematical probability expressions are obtained for the contagious distribution in this paper that enable one to apply the distribution to the data in practice. The moment's expressions of the contagious distribution can be readily applied by moments estimated by the data. An example of this application is published somewhere else [4] for case $S_{1,N}$.
The Poisson-stopped-sum-distribution has been mainly studied for the case that X's are discrete variables. This work has specific features in a view that

it develops concrete expressions of the Poisson-stopped-sum-distribution for the case that X's are continuous variables, and that it further gives the Binomial-stopped-sum-distribution for both cases that X-variables are discrete or continuous.

For the cases of the Poisson and the Binomial-stopped-sum-distributions with X-variable being discrete or continuous, moments expressions are calculated in two forms, one distinguishes the case of number variable $N = 0$ from $N \neq 0$ and the other does not. These two types of expressions may clearly show the structure of the contagious distribution and may avoid one from careless mistakes of the moments expressions when these are applied to practical data: sometimes the case of $N = 0$ is not taken into account for a mistake.

The reason that we consider the Binomial-stopped-sum-distribution is that, depending on the variable of N, the Binomial distribution may be more appropriate than the Poisson distribution, for instance, for the case that the size of N is relatively small. The size of N must suggest important structure of the data concerned.

In practical applications two types of continuous variables considered here are useful, one from Gamma distribution and the other from Normal distribution. The Gamma distribution may explain many data, because of having as usual constraint $X > 0$ as many data does. The Normal distribution may be significant in such a case that the data has a certain central standard value from its internal structure of the data, and that actual occurrence of X-variable appears symmetrically against this central standard value.

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