Existence of primitive PL-complex decomposition for lattice PL-figures

Existence of primitive PL-complex decomposition for lattice PL-figures

Tetsunori KUROGI and Osami YASUKURA

(Received 26 AUGUST 2003)

Abstract. The existence of primitive PL-complex decomposition is proved for any lattice PL-figures.

1. Introduction.

For a subset P of a coordinate plane $\mathbf{R}^2$, let ∂P be the boundary of P. By definition, *a PL-figure* is a compact subset P of $\mathbf{R}^2$ such that ∂P can be decomposed as *a PL-complex in* $\mathbf{R}^2$ *of dimension at most* 1. Note that *a PL-complex in* $\mathbf{R}^2$ is a set K of $\emptyset$, points, line segments (i.e. closed line segments of positive length), or triangles (i.e. closed triangular disks of positive area) in $\mathbf{R}^2$ satisfying the following two conditions:

(PLC. 1) If σ is an element of K, then $\partial\sigma$ is a union of certain elements of K. Or, equivalently:

If a line segment is an element of K, then its two ends are also elements of K. If a triangle is an element of K, then its three edges are also elements of K. And so is the ends of the edges;

(PLC. 2) If σ_1 and σ_2 are two elements of K, then $\sigma_1 \cap \sigma_2$ is a union of certain elements of K. Or, equivalently:

The intersection of a point in K and a line segment in K is empty or one of the two ends of the line segment. The intersection of a line segment in K and a triangle in K is empty, one of the three vertexes of the triangle (which is also an end of the line segment), or one of the three edges of the triangle (which is also the line segment itself); The intersection of two line segments in K is empty or a common end of them. The intersection of two triangles in K is empty, a common vertex of them, or a common edge of them.

By definition, *a PL-complex in* $\mathbf{R}^2$ *of dimension at most* 1 is a PL-complex in $\mathbf{R}^2$ consisting of $\emptyset$, points or line segments in $\mathbf{R}^2$. A subset K' of a PL-complex K in $\mathbf{R}^2$ is said to be *a subcomplex* if K' is also a PL-complex in $\mathbf{R}^2$. It can be proved that any PL-figure has a decomposition as a PL-complex in $\mathbf{R}^2$ consisting of points, line segments or triangles in $\mathbf{R}^2$ (see Corollary in §4).

More generally, PL-complex is defined. And Radó showed that any Riemann surface admits a PL-complex decomposition. On the other hand, this paper is concerned with a PL-figure of special type, called a lattice PL-figure.

A point in $\mathbf{R}^2$ is said to be *a lattice point* if all of its coordinates are integers. Let $\mathbf{Z}^2$ be the set of all lattice points in $\mathbf{R}^2$. By definition, *a lattice line segment* is a line segment in $\mathbf{R}^2$ such that its two ends are lattice points. *A primitive line segment* is a lattice line segment which contains no lattice point except the two ends. *A lattice triangle* is a triangle in $\mathbf{R}^2$ such that its three vertexes are lattice points. *A primitive triangle* is a lattice triangle which contains no lattice point except the three vertexes. *A primitive PL-complex* is a PL-complex in $\mathbf{R}^2$ consisting of lattice points, primitive line segments, or primitive triangles.

Then *a lattice PL-figure* is defined to be a compact subset P in $\mathbf{R}^2$ such that ∂P can be decomposed as a primitive PL-complex of dimension at most 1. Note that a lattice line segment or a lattice triangle is a lattice PL-figure. A lattice PL-figure is not assumed to be connected, and that it may have isolated lattice points (as 0-dimensional connected components) or local 1-dimensional parts. Moreover, various Euler number can be occured.

THEOREM. *Any lattice PL-figure P can be decomposed to a primitive PL-complex such that the primitive PL-complex of ∂P is a subcomplex.*

In the literature of Pick's theorem [3, 6], this fact was more or less assumed although it has not been proved completely (cf. [1, 4]). This paper fills the gap in the literature of Pick's theorem, in this general situation, by virtue of the notion of the distance between two compact subsets, combined with the existence of a lattice triangle in any lattice PL-figure. Note that the combined result is proved in the step B as well as the claims (B1) and (B5) in §3, by generalizing Sunada [5, pp.85-86, Proof of Theorem 1.9])'s proof of the existence of PL-complex decomposition for polygonal regions in $\mathbf{R}^2$ of Euler number 1.

To prove the theorem, the following proposition is needed: For $r > 0$, put $U_r(v) := \{p \in \mathbf{R}^2 | \text{dist.}(v,p) < r\}$, where dist.$(v,p)$ is the Euclidean distance between the two points v and p in $\mathbf{R}^2$. For a subset P of $\mathbf{R}^2$, let ext.P be the set of all exterior points of P, and int.P the set of all interior points of P. For $x, y \in \mathbf{R}^2$, put $xy := \{tx + (1-t)y | \ 0 \leq t \leq 1\}$ as the closed line segment with the two ends x, y. And put $xy^\circ := \{tx + (1-t)y | \ 0 < t < 1\}$ as the open line segment between x and y.

PROPOSITION. *Let P be a lattice PL-figure. Then:*

(i) *$\partial(\text{int.}P)$ can be decomposed to a primitive PL-complex which is a subcomplex of the primitive PL-complex of ∂P;*

(ii) *For any $v \in \partial(\text{int.}P) \cap \mathbf{Z}^2$, there exist $w, w' \in \mathbf{Z}^2$ such that $w \neq w'$, $vw, vw' \subseteq \partial(\text{int.}P)$ and that vw and vw' are primitive line segments;*

(iii) *If int.$P \neq \emptyset$, then there exist finite numbers, say, k of primitive line segments $L_i := x_i y_i$ $(i = 1, \cdots, k)$ such that $\partial(\text{int.}P) = \cup_{i=1}^k x_i y_i$ and that*

$$x_iy_i \cap x_jy_j = \emptyset,\ \{x_i\}\ \textit{or}\ \{y_i\}$$

for $i \neq j$*;*

(iv) *For each* $z \in L_i^\circ := x_iy_i^\circ$*, there exists* $r > 0$ *such that all conected components of* $U_r(z)\backslash L_i$ *are the following two subsets:*

$$U_r(z) \cap \text{ext}.P \quad \textit{and} \quad U_r(z) \cap \text{int}.P.$$

2. Proof of Proposition.

(i) If $\text{int}.P = \emptyset$, then there is nothing to do. Assume that $\text{int}.P \neq \emptyset$. Note that $\partial(\text{int}.P) \subseteq \partial P$. Since ∂P is decomposed as a primitive PL-complex of dimension at most 1, there exist primitive line segments $L_i := x_iy_i$ $(i = 1, \cdots, k')$ and lattice points p_j $(j = 1, \cdots, m)$ such that

$$\partial P = L_1 \cup \cdots \cup L_{k'} \cup \{p_1, \cdots, p_m\}, \tag{1}$$

where $p_j \notin L_i$ $(i = 1, \cdots, k';\ j = 1, \cdots, m)$, $p_j \neq p_{j'}$ $(j \neq j')$, and that

$$L_i \cap L_{i'} = \emptyset,\ \{x_i\},\ \text{or}\ \{y_i\}\ \text{if}\ i \neq i'.$$

It is then claimed that $L_i \subseteq \partial(\text{int}.P)$ *if* $L_i^\circ \cap \partial(\text{int}.P) \neq \emptyset$. In this case, the set of all points and line segments of the primitive PL-complex of ∂P contained in $\partial(\text{int}.P)$ gives a primitive PL-complex decomposition of $\partial(\text{int}.P)$ as a subcomplex of the primitive PL-complex of ∂P.

Assume that $L_i^\circ \cap \partial(\text{int}.P) \neq \emptyset$. It is then claimed that $L_i^\circ \subseteq \partial(\text{int}.P)$ (Then $L_i \subseteq \partial(\text{int}.P)$, by taking the closure, as required). Take $x \in L_i^\circ \cap \partial(\text{int}.P)$. It is enough to prove that $y \in \partial(\text{int}.P)$ *for any* $y \in L_i^\circ\backslash\{x\}$*:* Put $B_i := \partial P\backslash L_i^\circ$. Then $B_i = \cup_{i' \neq i} L_{i'} \cup \{p_1, \cdots, p_m\} \cup \{x_i, y_i\}$, because of $L_i = L_i^\circ \cup \{x_i, y_i\}$. And $xy \cap B_i = \emptyset$, because of $xy \subseteq L_i^\circ$. Put $s_y := \min\{\text{dist}.(p, b) |\ p \in xy, b \in B_i\}$, which is positive since B_i and xy are compact non-intersecting subsets in $\mathbf{R}^2$. Hence, one has that

$$U_{s_y}(p) \cap B_i = \emptyset \text{ for all } p \in xy. \tag{2}$$

For each $t > 0$, $U_t(x) \cap \text{int}.P \neq \emptyset$, because of $x \in \partial(\text{int}.P)$. Since $U_t(x) \cap \text{int}.P$ is an open subset of $\mathbf{R}^2$, there exists $z_t \in U_t(x) \cap \text{int}.P$ such that $z_t - x$ and $y - x$ are linearly independent. Put $w_t := z_t + y - x$. Then $w_t \in U_t(y)$. It is then claimed that $w_t \in \text{int}.P$ *if* $0 < t < s_y$*:* In this case, $U_t(y) \cap \text{int}.P \neq \emptyset$, so that $y \in \partial(\text{int}.P)$, as required. Assume that the claim does not hold. Then there exists $t \in \mathbf{R}$ such that $0 < t < s_y$ and $w_t \notin \text{int}.P$. Because of $z_t \in \text{int}.P$, there exists $q_t \in z_tw_t$ such that $q_t \in \partial(\text{int}.P)$. Take $0 \leq r \leq 1$ such that

$$q_t = rz_t + (1-r)w_t = z_t + (1-r)(y-x). \tag{3}$$

Put $p_t := rx+(1-r)y \in xy$. Then $q_t - p_t = z_t - x$, so that $q_t \in U_t(p_t) \subseteq U_{s_y}(p_t)$, so that $q_t \notin B_i$ by (2). Then $q_t \in \partial(\text{int}.P)\backslash B_i \subseteq \partial P\backslash B_i = L_i^\circ = x_i y_i^\circ \supset xy$, so that there exists a real number s such as $q_t = (1-s)x + sy$. Combined with (3), one has that $x - z_t = (1-r-s)(y-x)$, that contradicts with the choice of z_t.

(ii) For $v \in \partial(\text{int}.P) \cap \mathbf{Z}^2$, it is claimed that *there exists a primitive line segment* $vw \in \partial(\text{int}.P)$*:* If not, v is isolated in $\partial(\text{int}.P)$. Since $\partial(\text{int}.P)\backslash\{v\}$ is compact, there exists $r > 0$ such that

$$U_r(v) \cap (\partial(\text{int}.P)\backslash\{v\}) = \emptyset.$$

Because of $v \in \partial(\text{int}.P)$, there exist $x \in U_r(v)\cap\text{int}.P$ and $y \in U_r(v)\cap\text{ext}.(\text{int}.P)$. Since $U_r(v)\backslash\{v\}$ is connected, there is a continuous curve $c : [0,1] \to U_r(v)\backslash\{v\}$ such that $c(0) = x$ and $c(1) = y$. Then

$$\begin{aligned} \emptyset \neq c([0,1]) \cap \partial(\text{int}.P) &\subseteq (U_r(v)\backslash\{v\}) \cap \partial(\text{int}.P) \\ &\subseteq U_r(v) \cap (\partial(\text{int}.P)\backslash\{v\}). \end{aligned}$$

Hence, $U_r(v) \cap (\partial(\text{int}.P)\backslash\{v\}) \neq \emptyset$, that is a contradiction, as required.

It is then claimed that *there is a primitive line segment* vw' *in* $\partial(\text{int}.P)$ *such that* $w' \neq w$*:* Put $L_1 := vw$ and $C := \cup_{i=2}^{k'} L_i \cup \{p_1, \cdots, p_m\} \supseteq \partial P\backslash L_1$ in the equation (1). Assume that the claim does not hold. Then $v \notin C$, so that there is $r > 0$ such that

$$U_r(v) \cap C = \emptyset.$$

By $v \in \partial(\text{int}.P)$, there exist $x \in U_r(v)\cap\text{int}.P$ and $y \in U_r(v)\cap\text{ext}.(\text{int}.P)$. Since $U_r(v)\backslash vw$ is connected, there exists a continuous curve $c : [0,1] \to U_r(v)\backslash vw$ such that $c(0) = x$ and $c(1) = y$. Then $\emptyset \neq c([0,1]) \cap \partial(\text{int}.P) \subseteq (U_r(v)\backslash vw) \cap \partial(\text{int}.P) = U_r(v) \cap (\partial(\text{int}.P)\backslash L_1) \subseteq U_r(v) \cap C$, so that $U_r(v) \cap C \neq \emptyset$, that is a contradiction, as required.

(iii) It follows from the proof of (i) and the assertion (ii) that the set of all line segments of the primitive PL-complex of ∂P contained in $\partial(\text{int}.P)$ gives a primitive PL-complex decomposition of $\partial(\text{int}.P)$ as a subcomplex of the primitive PL-complex of ∂P.

(iv) Put $r_i := \min\{\text{dist}.(z,b) |\ b \in \partial P\backslash x_i y_i^\circ\}$, which is positive by $z \in x_i y_i^\circ$. Then $\partial P \cap (U_{r_i}(z)\backslash x_i y_i^\circ) = \emptyset$. Put $r := \min(r_i, \text{dist}.(z,x_i), \text{dist}.(z,y_i)\} > 0$. Then $U_r(z)\backslash x_i y_i = U_r(z)\backslash x_i y_i^\circ$ consists of two connected components, in which there exist $x \in \text{ext}.P$ and $y \in \text{int}.P$ by $z \in x_i y_i \subseteq \partial P$. Then the connected component of $U_r(z)\backslash x_i y_i$ containing x (resp. y) is equal to $U_r(z) \cap \text{ext}.P$ (resp. $U_r(z) \cap \text{int}.P$), as well as the proof of (ii). □

3. Proof of Theorem.

(Step O) For a compact subset P' in $\mathbf{R}^2$, let $A(P')$ be the area of P'.

(Step A) When $A(P) = 0$: $\text{int}.P = \emptyset$, so that $P = \text{int}.P \cup \partial P = \partial P$, which has a primitive PL-complex decomposition of dimension at most 1, by the definition of P.

(Step B) When $A(P) \neq 0$, it is claimed that *there exists a lattice triangle contained in P such that at least one vertex is contained in ∂P*. In fact:

(B0) Because of $\text{int}.P \neq \emptyset$ and $\text{int}.P \neq \mathbf{R}^2$, one has that $\partial(\text{int}.P) \neq \emptyset$.

(B1) *There exist $v \in \partial(\text{int}.P) \cap \mathbf{Z}^2$ and a line ℓ in $\mathbf{R}^2$ such that $v \in \ell$, $\ell \cap \partial(\text{int}.P) = \{v\}$, and that $\partial(\text{int}.P)\backslash\{v\}$ is contained in one of two connected components of $\mathbf{R}^2\backslash\ell$.*

In fact, put $d' := \max.\{\text{dist}.(v', v'') |\ v', v'' \in \partial(\text{int}.P) \cap \mathbf{Z}^2\}$. Then $d' > 0$ by (B0) and Proposition (ii), (iii). And there exist $v, v' \in \partial(\text{int}.P) \cap \mathbf{Z}^2$ such that $\text{dist}.(v, v') = d'$. By Proposition (ii), there exists a primitive line segment L_i in $\partial(\text{int}.P)$ such that v is one of two ends of L_i. Let ℓ be a line through v perpendicular to $v'v$. Then $v' \notin \ell$. Let O', O be the two connected components of $\mathbf{R}^2\backslash\ell$ such that $v' \in O'$. Then $\partial(\text{int}.P) \cap \mathbf{Z}^2 \subseteq O' \cup \{v\}$. If not, there exists $v'' \in \partial(\text{int}.P) \cap \mathbf{Z}^2 \cap (O \cup (\ell\backslash\{v\}))$. In this case, $\text{dist}.(v'', v') > \text{dist},(v, v') = d'$, that is a contradiction. By Proposition (iii), $\partial(\text{int}.P)$ is contained in the convex hull of $\partial(\text{int}.P) \cap \mathbf{Z}^2$. And $O' \cup \{v\}$ is convex. Hence, $\partial(\text{int}.P) \subseteq O' \cup \{v\}$.

(B2) Let $vw_1, \cdots, vw_k$ be distinct primitive line segments in P with v as one of its two ends $(i = 1, \cdots, k)$, such that k is maximal with this property, where $vw_1, \cdots, vw_k$ are anti-clockwisely ordered around the point v.

(B3) Then $k \geq 2$ by Definition-Proposition 3.1 (ii). Identifying $\mathbf{R}^2$ with the complex number field $\mathbf{C}$, put $r_v := |v|$ and $\theta_v := \arg(v)$ for $0 \neq v \in \mathbf{R}^2$. By (B1) and (B2), there exist $\theta_{w_1-v} < \cdots < \theta_{w_k-v}$ such that $\theta_{w_k-v} - \theta_{w_1-v} < \pi$. For $i \in \{1, \cdots, k-1\}$, put $\angle w_i v w_{i+1} := \{v + re^{\sqrt{-1}\theta} |\ r \geq 0,\ \theta_{w_i-v} \leq \theta \leq \theta_{w_{i+1}-v}\}$. And put $\angle w_k v w_1 := \{v + re^{\sqrt{-1}\theta} |\ r \geq 0,\ \theta_{w_k-v} \leq \theta \leq \theta_{w_1-v} + 2\pi\}$. Then

$$\mathbf{R}^2 = \cup_{i=1}^{k} \angle w_i v w_{i+1},$$

where $i + 1$ denotes 1 if $i = k$. Put

$$r := \min.\{\text{dist}.(v, w') |\ w' \in \partial(\text{int}.P)\backslash(\cup_{i=1}^{k} vw_i^{\circ} \cup \{v\})\}.$$

Then $r > 0$, because of $v \notin \partial(\text{int}.P)\backslash(\cup_{i=1}^{k} vw_i^{\circ} \cup \{v\})$. In this case,

$$U_r(v) \cap \partial(\text{int}.P)\backslash \cup_{i=1}^{k} vw_i = \emptyset. \tag{4}$$

By $v \in \partial(\text{int}.P)$, $U_r(v) \cap \text{int}.P \neq \emptyset$. Hence, there exists $i \in \{1, \cdots, k\}$ such that

$$((U_r(v) \cap \angle w_i v w_{i+1})\backslash(vw_i \cup vw_{i+1})) \cap \text{int}.P \neq \emptyset. \tag{5}$$

Because of the equation (4) and $((U_r(v)\cap\angle w_ivw_{i+1})\backslash(vw_i\cup vw_{i+1}))\cap\partial(\text{int}.P) \subseteq U_r(v)\cap\partial(\text{int}.P)\backslash\cup_{i=1}^k vw_i$, one has that

$$((U_r(v)\cap\angle w_ivw_{i+1})\backslash(vw_i\cup vw_{i+1}))\cap\partial(\text{int}.P)=\emptyset. \qquad (6)$$

It is then claimed that

$$(U_r(v)\cap\angle w_ivw_{i+1})\backslash(vw_i\cup vw_{i+1})\subseteq \text{int}.P. \qquad (7)$$

In fact, by (5), take $x\in((U_r(v)\cap\angle w_ivw_{i+1})\backslash(vw_i\cup vw_{i+1}))\cap\text{int}.P$. Note that $(U_r(v)\cap\angle w_ivw_{i+1})\backslash(vw_i\cup vw_{i+1})$ is connected. If there is $x'\in((U_r(v)\cap\angle w_ivw_{i+1})\backslash(vw_i\cup vw_{i+1}))\cap\text{ext}.(\text{int}.P)$, then there is a continuous curve

$$c:[0,1]\to(U_r(v)\cap\angle w_ivw_{i+1})\backslash(vw_i\cup vw_{i+1})$$

such that $c(0)=x$ and $c(1)=x'$, so that $c([0,1])\cap\partial(\text{int}.P)\neq\emptyset$, that contradicts with the equation (6). Hence, $((U_r(v)\cap\angle w_ivw_{i+1})\backslash(vw_i\cup vw_{i+1}))\cap\text{ext}.(\text{int}.P)=\emptyset$. Combined with the equation (6), one has the equation (7), as required.

(B4) In (7), it is claimed that $i\neq k$.

In fact, if $i=k$, then the half line ℓ' for O (in (B1)) from v perpendicular to ℓ contains an element of int.P. So does $\ell'\backslash\{v\}$. Since $\ell'\backslash\{v\}$ is not bounded, $\emptyset\neq(\ell'\backslash\{v\})\cap(\mathbf{R}^2\backslash P)\subseteq(\ell'\backslash\{v\})\cap\text{ext}.(\text{int}.P)$. By (7), $(\ell'\backslash\{v\})\cap\text{int}.P\neq\emptyset$. Hence, $\emptyset\neq(\ell'\backslash\{v\})\cap\partial(\text{int}.P)\subseteq O$, which contradicts with (B1).

(B5) Let $\triangle w_ivw_{i+1}$ be the convex hull of the set $\{w_i,v,w_{i+1}\}$. By (B4), $\triangle w_ivw_{i+1}\subset\angle w_ivw_{i+1}$. It is claimed that $\text{int}.(\triangle w_ivw_{i+1})\cap P\cap\mathbf{Z}^2=\emptyset$.

In fact, assume that the claim does not hold, and put

$$s:=\max\ \{\text{dist}.(w_iw_{i+1},z')|\ z'\in\text{int}.(\triangle w_ivw_{i+1})\cap P\cap\mathbf{Z}^2\}>0.$$

Take $z\in\text{int}.(\triangle w_ivw_{i+1})\cap P\cap\mathbf{Z}^2$ such that $\text{dist}.(v,z)=s$. By (B2), $vz\not\subseteq P$, so that $\emptyset\neq vz^\circ\cap(\mathbf{R}^2\backslash P)\subseteq vz^\circ\cap\text{ext}.(\text{int}.P)$. By (7), $vz^\circ\cap\text{int}.P\neq\emptyset$. Hence, $vz^\circ\cap\partial(\text{int}.P)\neq\emptyset$. Take $w\in vz^\circ\cap\partial(\text{int}.P)$. Then $\text{dist}.(w_iw_{i+1},w)>s$. By Proposition (iii), there exists a primitive line segment $x_jy_j\subseteq\partial(\text{int}.P)$ such that $w\in x_jy_j$. Then

$$\max.(\text{dist}.(w_iw_{i+1},x_j),\text{dist}.(w_iw_{i+1},y_j))\geq\text{dist}.(w_iw_{i+1},w)>s.$$

Assume that $\text{dist}.(w_iw_{i+1},x_j)>s$. Then $x_j\notin\text{int}.(\triangle w_ivw_{i+1})\cup w_iw_{i+1}$. And $x_j\notin vw_i^\circ\cup vw_{i+1}^\circ$ because vw_i and vw_{i+1} are primitive. By (B2), $x_j\neq v$. Then $x_j\notin\triangle w_ivw_{i+1}$, so that $x_jy_j\cap vw_{i'}\neq\emptyset$, transversally, for $i'=i$ or $i+1$. Put $\{w'\}:=x_jy_j\cap vw_{i'}\subseteq x_jy_j^\circ$. By Proposition (iv), there exists a sufficiently small $r'>0$ such that the two connected components O,O' of $U_{r'}\backslash x_jy_j$ satisfy that $O\subseteq\text{int}.P$ and $O'\subseteq\text{ext}.P$. However, $P\cap O'\supseteq vw_{i'}\cap O'\neq\emptyset$, by the transversality, which is a contradiction. In the case when $\text{dist}.(w_iw_{i+1},y_j)>s$, one also has a contradiction.

(B6) $\text{int.}(\triangle w_i v w_{i+1}) \cap \partial(\text{int.}P) = \emptyset$.

In fact, assume that the assertion does not hold. Then there exists $w \in \partial(\text{int.}P)$ such that $w \in \text{int.}(\triangle w_i v w_{i+1})$. By Proposition (iii), there exists a primitive line segment $x_j y_j \subseteq \partial(\text{int.}P)$ such that $w \in x_j y_j$. By (B5), $x_j, y_j \notin \text{int.}(\triangle w_i v w_{i+1})$. Since vw_i and vw_{i+1} are primitive, $x_j, y_j \notin vw_i^\circ \cup vw_{i+1}^\circ$. By (B2), $x_j, y_j \notin \{v\}$. If $x_j \in w_i w_{i+1}$ and $y_j \in w_i w_{i+1}$, then $w \in w_i w_{i+1}$, which does not intersect with $\text{int.}(\triangle w_i v w_{i+1})$. Hence, $x_j \notin \triangle w_i v w_{i+1}$ or $y_j \notin \triangle w_i v w_{i+1}$. Then one has a contradiction as well as (B5).

(B7) $\triangle w_i v w_{i+1} \subseteq P$.

In fact, assume that there exists $w \in \text{int.}(\triangle w_i v w_{i+1}) \backslash \text{int.}P$. By (7), there exists $u \in \text{int.}(\triangle w_i v w_{i+1}) \cap \text{int.}P$. Then $\emptyset \neq uw \cap \partial(\text{int.}P)$. Since $\text{int.}(\triangle w_i v w_{i+1})$ is convex, one has that $uv \subseteq \text{int.}(\triangle w_i v w_{i+1})$, so that

$$\text{int.}(\triangle w_i v w_{i+1}) \cap \partial(\text{int.}P) \neq \emptyset,$$

which contradicts with (B6). Hence, $\text{int.}(\triangle w_i v w_{i+1}) \subseteq \text{int.}P$. By taking the closure of the both sides, one has the required result.

(Step C) Assume that $A(P) \neq 0$. Let $S(P)$ be the finite set of all lattice PL-figure P' contained in P. By (Step B), $S_1(P) := \{P' \in S(P) |\ A(P') > 0\} \neq \emptyset$. Then

$$\alpha_P := \min\{A(P') |\ P' \in S_1(P)\}.$$

is a well-defined positive real number. It is claimed that *any* $P' \in S(P)$ *can be decomposed to a primitive PL-complex* K' *such that the primitive PL-complex of* $\partial P'$ *is a subcomplex of* K'.

(C0) For any $P' \in S(P)$, let $n_{P'}$ be a unique integer such that

$$(n_{P'} - 1)\alpha_P \leq A(P') < n_{P'} \alpha_P.$$

Then the proof of the above claim is given by the induction on $n_{P'}$ as follows:

(C1) When $n_{P'} = 1$: $A(P') = 0$ and $\text{int.}P' = \emptyset$, so that $P' = \partial P'$ is decomposed as a primitive PL-complex at most 1, by the definition of a lattice PL-figure P', as required.

(C2) Assume that the claim holds for any $P' \in S(P)$ such that $n_{P'} \leq k - 1$ for a fixed integer $k \geq 2$. Consider any $P' \in S(P)$ such that $n_{P'} = k$. Then $A(P') \neq 0$. By (Step B), one concludes that there exist $v \in \partial P'$ and a lattice triangle $\triangle v_1 v v_2 \in S_1(P')$.

(C3) *There exists a primitive triangle* $\triangle u_1 u_2 u_3 \in S_1(\triangle v_1 v v_2) \subseteq S_1(P')$: In fact, if $\triangle v_1 v v_2$ is primitive, put $u_i := v_i$ $(i = 1, 2)$. If $\triangle v_1 v v_2$ is not primitive, then there exists a lattice point $u_1 \in \triangle v_1 v v_2$ such that $u_1 \neq v, v_1, v_2$. Then $u_1 \notin vv_1$ or $u_1 \notin vv_2$. If $u_1 \notin vv_1$, put $u_2 := v_1$ and $u_3 := v_2$. Then

$$A(\triangle v_1 v v_2) = A(\triangle u_1 v u_2) + A(\triangle u_1 v u_3) + A(\triangle u_1 u_2 u_3),$$

so that $A(\triangle u_1vu_3)+A(\triangle u_1u_2u_3)=A(\triangle v_1vv_2)-A(\triangle u_1vu_2)>0$. Hence, $A(\triangle u_1vu_3)+A(\triangle u_1u_2u_3)\geq \alpha_P$. Then

$$A(\triangle u_1vu_2)\leq A(\triangle v_1vv_2)-\alpha_P\leq (k-1)\alpha_P,$$

so that $\triangle u_1vu_2$ admits a primitive PL-complex decomposition such that the primitive PL-complex of $\partial(\triangle u_1vu_2)$ is a subcomplex, by the assumption of induction. In particular, it contains a primitive triangle.

(C4) Put $Q := P'\backslash \text{int.}(\triangle u_1u_2u_3)$ and $C := \partial P'\cap\partial(\triangle u_1u_2u_3)$. Note that $\partial(\triangle u_1u_2u_3)=u_1u_2\cup u_2u_3\cup u_3u_1$, so that

$$C=\{u_i\},\{u_i,u_j\},\{u_1,u_2,u_3\},\{u_i\}\cup u_ju_k, u_iu_j\cup u_iu_k, \text{or } \partial(\triangle u_1u_2u_3),$$

where $\{i,j,k\}=\{1,2,3\}$. For any subset P'' of $\mathbf{R}^2$, put $\text{cl.}(P'') := P''\cup\partial P''$. Then

$$\partial Q=\text{cl.}(\partial P'\backslash C)\cup\text{cl.}(\partial(\triangle u_1u_2u_3)\backslash C),$$

which admits a primitive PL-complex decomposition of dimension at most 1. Hence, Q is a lattice PL-figure with the area

$$A(Q)=A(P')-A(\triangle u_1u_2u_3)\leq (k-1)\alpha_P.$$

By the induction assumption, Q admits a primitive PL-complex decomposition such that the primitive PL-complex of ∂Q is a subcomplex. Then

$$P'=Q\cup\triangle u_1u_2u_3$$

can be decomposed to a primitve PL-complex K' by the union of the primitive PL-complex decompositions of Q and $\triangle u_1u_2u_3$ such that the primitive PL-complex of $\partial P'$ is a subcomplex of K'. □

4. Concluding Remarks

REMARK. The statements and the proof of Proposition and Theorem hold also when the definition of "lattice points" $\mathbf{Z}^2$ is replaced by any subset Z in $\mathbf{R}^2$ such that $D\cap Z$ is a finite set for any bounded subset D in $\mathbf{R}^2$.

In particular, one has the following result:

COROLLARY. *Any PL-figure P can be decomposed to a PL-complex consisting of points in the set Z of all "points" in the PL- complex of ∂P, line segments such that their two ends are contained in Z, or triangles such that their three vertexes are contained in Z.*

Proof of Corollary from Theorem: Note that the set Z of all "points" in the PL-complex of ∂P satisfy the condition in the Remark, and that P is a "lattice" PL-figure with respect to Z. Hence, the assertion follows from the Theorem. □

Note that Proposition, Theorem, Remark and Corollary are used in [2] as Definition-Proposition 3.1, Theorem 3.2, Remark 3.4 and Proposition 5.2.

References

[1] R.W. Gaskell, M.S. Klamkin and P. Watson, Triangulations and Pick's Theorem, *Mathematics Magazine* **49**-1 (1976) 35–37.

[2] Tetsunori Kurogi and Osami Yasukura, From Homma's theorem to Pick's theorem, *preprint*.

[3] Ivan Niven and H.S. Zuckerman, Lattice points and polygonal area, *American Mathematical Monthly* **74** (December 1967) 1195–1200.

[4] Ira Rosenholtz, Calculating Surface Areas from a Blueprint, *Mathematics Magagine* **52**-4 (September 1979) 252–256.

[5] Koji Shiga and Toshikazu Sunada, *Mathematics presented to high school students III*, Iwanami-Shoten, Tokyo, 1996 (in Japanese).

[6] Dale E. Varberg, Pick's theorem revised, *American Mathematical Monthly* **92** (October 1985) 584–587.

Fukui University, Fukui-shi, 910-8507, Japan