A relation between super Weyl groupoids and Coxeter groupoids

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A relation between super Weyl groupoids and Coxeter groupoids

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Abstract. In this paper, we study a relation between super Weyl groupoids defined by Sergeev and Veselov [9] and Coxeter groupoids by Heckenberger and Yamane [3]. As an application, we provide generators and defining relations for the super Weyl groupoids.

1. Introduction

In the representation theory of basic classical Lie superalgebras, certain reflections, called odd reflections, play important roles. An odd reflection is defined as a transformation between basis of a root system (see e.g. [10]), and it is not necessarily extended to a linear transformation on the dual space of the Cartan subalgebra containing the root system. Hence, it is not obvious what kind of algebraic system should be considered in order to treat ordinary (real) and odd reflections at the same time.

In [3], I. Heckenberger and H. Yamane proposed the notion of a Coxeter groupoid from the viewpoint of structures of Nichols algebras and basic classical Lie superalgebras. On the other hand, in [9], A. N. Sergeev and P. Veselov introduced the super Weyl groupoid associated with a basic classical Lie superalgebra, and show that the representation rings of the Lie superalgebras can be regarded as invariants of the groupoid.

The main purpose of this article is to give a relation between the super Weyl groupoids associated with the basic classical Lie superalgebras and Coxeter groupoids. Since a super Weyl groupoid is defined as the disjoint union of the Weyl group of the corresponding Lie superalgebra and a certain groupoid, we concentrate on the groupoid part. In this paper, we introduce the notion of an involutive Coxeter groupoid and its

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extension. The groupoid part of a super Weyl groupoid is obtained as the extension of an involutive Coxeter groupoid. Moreover, the structure of the involutive Coxeter groupoid in a super Weyl groupoid can be described by means of (1) the direct product of Coxeter groupoids, (2) the semi-direct product groupoid constructed from an action of a Weyl group. As an application, we obtain generators and defining relations for the super Weyl groupoids.

Finally, we remark on an expected relation between a super Weyl groupoid and certain generalized Verma modules over the corresponding basic classical Lie superalgebra. It is well-known that the embedding structure of Verma modules over a finite dimensional simple Lie algebra can be described by its Weyl group. At least in the simplest case, one can observe a similar relation between generalized Verma modules over $\mathfrak{sl}(2,1)$ and the super Weyl groupoid of type $A(1,0)$.

This paper is organized as follows: In Section 2, after giving some basic definitions in the theory of groupoids, we recall the definitions of a super Weyl groupoid and a Coxeter groupoid and give examples of them. In Section 3, we introduce the notion of an involutive Coxeter groupoid and its extension, and show that the groupoid part of a super Weyl groupoid is isomorphic to the extension of an involutive Coxeter groupoid. In Section 4, we explicitly describe the super Weyl groupoids of types $A(m,n)$, $B(m,n)$, $C(n)$ and $D(m,n)$ in terms of the Weyl groups of simple Lie algebras. In Section 5, we compare the embedding structure of generalized Verma modules over $\mathfrak{sl}(2,1)$ and the super Weyl groupoid of type $A(1,0)$. In Appendix, we analyze the structure of the atypical integrable representations over $\mathfrak{sl}(2,1)$ by means of generalized Verma modules.

Throughout this paper, we denote the cardinality of a set A by $\sharp A$, and the disjoint union of sets A and B by $A \sqcup B$. The vector spaces are over the complex number field $\mathbb{C}$ unless otherwise stated.

2. Super Weyl groupoids and Coxeter groupoids

2.1. Preliminaries

A super Weyl groupoid is defined to be the semi-direct product groupoid of a group and a groupoid. Here, we give the definition of semi-direct product groupoids following [1].

Recall that a groupoid $\mathfrak{G}$ is a small category such that any morphism is an isomorphism. The set $\mathrm{Ob}(\mathfrak{G})$ of all objects is called base of $\mathfrak{G}$. Let

Γ be a group with the unit 1_Γ, and $\mathfrak{G}$ a groupoid. We first introduce the notion of a group action on a groupoid.

Definition 2.1. An action of Γ on $\mathfrak{G}$ is a collection of the following maps: For each $\gamma \in \Gamma$,

(1) $\gamma : \mathrm{Ob}(\mathfrak{G}) \to \mathrm{Ob}(\mathfrak{G})$ satisfying

$$(\gamma' \circ \gamma)(x) = \gamma'(\gamma(x)), \quad 1_\Gamma(x) = x \quad (\forall \gamma, \gamma' \in \Gamma,\ \forall x \in \mathrm{Ob}(\mathfrak{G})),$$

(2) $\gamma : \mathrm{Hom}_{\mathfrak{G}}(X, Y) \to \mathrm{Hom}_{\mathfrak{G}}(\gamma(X), \gamma(X))$ satisfying

$$(\gamma' \circ \gamma)(f) = \gamma'(\gamma(f)), \quad 1_\Gamma(f) = f \quad (\forall \gamma, \gamma' \in \Gamma,\ \forall f \in \mathrm{Hom}_{\mathfrak{G}}(X, Y)),$$
$$\gamma(g \circ f) = \gamma(g) \circ \gamma(f) \quad (\forall \gamma \in \Gamma,\ \forall f \in \mathrm{Hom}_{\mathfrak{G}}(X, Y),\ \forall g \in \mathrm{Hom}_{\mathfrak{G}}(Y, Z)).$$

By definition, it is easy to see that $\gamma(\mathbf{1}_X) = \mathbf{1}_{\gamma(X)}$.

Definition 2.2. The semi-direct product groupoid $\Gamma \ltimes \mathfrak{G}$ of Γ and $\mathfrak{G}$ is the groupoid defined by

(1) Objects: $\mathrm{Ob}(\Gamma \ltimes \mathfrak{G}) := \mathrm{Ob}(\mathfrak{G})$,

(2) Morphisms: $\mathrm{Hom}_{\Gamma \ltimes \mathfrak{G}}(X, Y) := \{(\gamma, f) | \gamma \in \Gamma,\ f \in \mathrm{Hom}_{\mathfrak{G}}(\gamma(X), Y)\}$,

(3) Composition:
$$(\delta, g) \circ (\gamma, f) := (\delta\gamma, g \circ \delta(f)), \tag{2.1}$$
where $(\gamma, f) \in \mathrm{Hom}_{\Gamma \ltimes \mathfrak{G}}(X, Y)$ and $(\delta, g) \in \mathrm{Hom}_{\Gamma \ltimes \mathfrak{G}}(Y, Z)$.

Suppose that a group Γ acts on a set M. We regard M as a *fine groupoid*, namely, the groupoid on M consisting only of identities. As a typical example of the semi-direct product groupoid, we have $\Gamma \ltimes M$. In fact, we will see in Section 2.4 that if Γ is a Weyl group, then $\Gamma \ltimes M$ is a Coxeter groupoid. For simplicity, set

$$[\gamma, m] := (\gamma, \mathbf{1}_{\gamma(m)}) \quad (\gamma \in \Gamma,\ m \in M). \tag{2.2}$$

Notice that $[\gamma, m] \in \mathrm{Hom}_{\Gamma \ltimes M}(m, \gamma(m))$. Under this setting, Definition 2.2 can be rewritten as follows:

Definition 2.3. The semi-direct product groupoid $\Gamma \ltimes M$ is defined by

(1) Objects: $\mathrm{Ob}(\Gamma \ltimes M) := M$.

(2) Morphisms: $\mathrm{Hom}_{\Gamma \ltimes M}(m, n) := \{[\gamma, m] | \gamma(m) = n\}$ for $m, n \in M$.

(3) Composition: $[\delta, n] \circ [\gamma, m] := [\delta\gamma, m]$, where $n = \gamma(m)$.

Finally, for later use, we recall the disjoint union and the direct product of two groupoids $\mathfrak{G}_i$ $(i = 1, 2)$. The disjoint union

$$\mathfrak{G}_1 \sqcup \mathfrak{G}_2 \tag{2.3}$$

is the groupoid defined by

(1) Objects: $\mathrm{Ob}(\mathfrak{G}_1 \sqcup \mathfrak{G}_2) := \mathrm{Ob}(\mathfrak{G}_1) \sqcup \mathrm{Ob}(\mathfrak{G}_2)$,

(2) Morphisms:

$$\mathrm{Hom}_{\mathfrak{G}_1 \sqcup \mathfrak{G}_2}(X, Y) := \begin{cases} \mathrm{Hom}_{\mathfrak{G}_i}(X, Y) & (X, Y \in \mathrm{Ob}(\mathfrak{G}_i)) \\ \emptyset & (\text{otherwise}) \end{cases}.$$

The direct product

$$\mathfrak{G}_1 \times \mathfrak{G}_2 \tag{2.4}$$

of the groupoids $\mathfrak{G}_i$ $(i = 1, 2)$ is defined as the product of categories:

(1) Objects: $\mathrm{Ob}(\mathfrak{G}_1 \times \mathfrak{G}_2) = \mathrm{Ob}(\mathfrak{G}_1) \times \mathrm{Ob}(\mathfrak{G}_2)$,

(2) Morphisms: for X_ℓ, $Y_\ell \in \mathrm{Ob}(\mathfrak{G}_\ell)$,

$$\mathrm{Hom}_{\mathfrak{G}_1 \times \mathfrak{G}_2}((X_1, X_2), (Y_1, Y_2)) := \{(f_1, f_2) | f_\ell \in \mathrm{Hom}_{\mathfrak{G}_\ell}(X_\ell, Y_\ell)\}$$

(3) Composition: $(g_1, g_2) \circ (f_1, f_2) := (g_1 \circ f_1, g_2 \circ f_2)$.

In Section 4.1, we will see that if both of $\mathfrak{G}_i$ are Coxeter groupoids, then $\mathfrak{G}_1 \times \mathfrak{G}_2$ is also a Coxeter groupoid.

2.2. Definition of super Weyl groupoids

In order to define super Weyl groupoids, we first introduce notation for Lie superalgebras. Let $\mathfrak{g}$ be a basic classical Lie superalgebra, and $\mathfrak{h}$ a Cartan subalgebra of $\mathfrak{g}$. Let $(\cdot, \cdot)$ be a non-degenerate even supersymmetric invariant bilinear form on $\mathfrak{g}$ and $\xi : \mathfrak{h} \to \mathfrak{h}^*$ the linear map defined by $\langle \xi(h), h' \rangle = (h, h')$. We introduce a non-degenerate bilinear form $(\cdot, \cdot)$ on $\mathfrak{h}^*$ as $(\lambda, \mu) = (\xi^{-1}(\lambda), \xi^{-1}(\mu))$. Let Δ be the root system of $\mathfrak{g}$, $\{\alpha_i\}_{i \in I}$ the set of the simple roots with an index set I and $Q := \sum_{i \in I} \mathbb{Z}\alpha_i$ the root lattice. Denote the sets of the even and the odd roots by $\Delta_{\bar{0}}$ and $\Delta_{\bar{1}}$. The subset $\Delta_{\mathrm{iso}} = \{\alpha \in \Delta_{\bar{1}} | (\alpha, \alpha) = 0\}$ of $\Delta_{\bar{1}}$ is the set of the isotropic odd roots. Put $\Delta^+ := \Delta \cap Q^+$, $\Delta_{\bar{0}}^+ := \Delta_{\bar{0}} \cap Q^+$, $\Delta_{\bar{1}}^+ := \Delta_{\bar{1}} \cap Q^+$ $\Delta_{\mathrm{iso}}^+ := \Delta_{\mathrm{iso}} \cap Q^+$, where $Q^+ := \sum_{i \in I} \mathbb{Z}_{\geq 0}\alpha_i$.

For $\gamma \in \Delta_{\bar{0}}$, define the real reflection $r_\gamma \in \mathrm{GL}(\mathfrak{h}^*)$ by $r_\gamma(\lambda) := \lambda - \langle \lambda, \gamma^\vee \rangle \gamma$, where $\gamma^\vee := \dfrac{2\xi^{-1}(\gamma)}{(\gamma, \gamma)}$ is the coroot of γ. Let W be the Weyl group of $\mathfrak{g}$, namely, W is the subgroup of $\mathrm{GL}(\mathfrak{h}^*)$ generated by all real reflections r_γ ($\gamma \in \Delta_{\bar{0}}$). We define the groupoid $\mathfrak{I}$ as follows:

(1) Objects: $\mathrm{Ob}(\mathfrak{I}) = \Delta_{\mathrm{iso}}$,

(2) Morphisms: for α, $\beta \in \Delta_{\mathrm{iso}}$,

$$\mathrm{Hom}_{\mathfrak{I}}(\alpha, \beta) := \begin{cases} \{\mathbf{1}_\alpha\} & (\beta = \alpha) \\ \{\mathbf{t}_\alpha\} & (\beta = -\alpha) \\ \emptyset & (\text{otherwise}) \end{cases} .$$

By definition, $\mathbf{t}_{-\alpha} \circ \mathbf{t}_\alpha = \mathbf{1}_\alpha$, $\mathbf{t}_\alpha \circ \mathbf{t}_{-\alpha} = \mathbf{1}_{-\alpha}$, and thus, $\mathbf{t}_{\pm\alpha}^{-1} = \mathbf{t}_{\mp\alpha}$.

Lemma 2.1. *The Weyl group W acts on $\mathfrak{I}$ as follows:*

$$w : \alpha \mapsto w(\alpha), \quad \mathbf{1}_\alpha \mapsto \mathbf{1}_{w(\alpha)}, \quad \mathbf{t}_\alpha \mapsto \mathbf{t}_{w(\alpha)} \quad (w \in W,\ \alpha \in \Delta_{\mathrm{iso}})$$

Then, one can consider the semi-direct product groupoid of W and $\mathfrak{I}$:

$$\mathfrak{W} := W \ltimes \mathfrak{I}. \tag{2.5}$$

We remark that

$$\mathrm{Hom}_{\mathfrak{W}}(\alpha, \beta) = \begin{cases} \{(w, \mathbf{1}_{w(\alpha)})\} & (\beta = w(\alpha) \text{ if } \exists w \in W) \\ \{(w, \mathbf{t}_{w(\alpha)})\} & (\beta = -w(\alpha) \text{ if } \exists w \in W) \\ \emptyset & (\text{otherwise}) \end{cases} . \tag{2.6}$$

Definition 2.4 ([9])**.** The super Weyl groupoid associated to a basic classical Lie superalgebra $\mathfrak{g}$ is the disjoint union $W \sqcup \mathfrak{W}$ of the Weyl group W of $\mathfrak{g}$ and the groupoid $\mathfrak{W}$ (see (2.3)). Here, we regard W as a groupoid with the base consisting of one point.

Since the structure of the Weyl group W is well-known ([5]), we concentrate on the groupoid part $\mathfrak{W}$.

Remark 2.1. If we regard Δ_{iso} as a fine groupoid, then the semi-direct product $W \ltimes \Delta_{\mathrm{iso}}$ is a subgroupoid of $\mathfrak{W}$. Since $\sharp\mathfrak{I} = 2 \times \sharp\Delta_{\mathrm{iso}}$, we have

$$\sharp\mathfrak{W} = 2 \times \sharp(W \ltimes \Delta_{\mathrm{iso}}). \tag{2.7}$$

In Subsection 2.4, we will show that $W \ltimes \Delta_{\mathrm{iso}}$ is a Coxeter groupoid.

Throughout this paper, we assume that Δ_{iso} is not empty, namely, $\mathfrak{g}$ is not a finite dimensional simple Lie algebra or the simple Lie superalgebra of type $B(0, n)$. For the classification of the basic classical Lie superalgebras, see [5].

Now, we look at an example of $\mathfrak{W}$ in the case where $\mathfrak{g}$ is of type $A(1, 0)$.

Example 2.1. A Cartan matrix of type $A(1, 0)$ is given by

$$\begin{pmatrix} 2 & -1 \\ -1 & 0 \end{pmatrix}. \tag{2.8}$$

Here, α_1 and α_2 are simple roots, and $\alpha := \alpha_1 + \alpha_2$. We have $\Delta^+_{\bar{0}} = \{\alpha_1\}$, $\Delta^+_{\bar{1}} = \{\alpha_2, \alpha\}$, $\Delta^+_{\mathrm{iso}} = \Delta^+_{\bar{1}}$ and $W = \{1, r\}$, where r denotes a unique even reflection ($r = r_{\alpha_1}$). Since $\sharp\mathfrak{I} = 2 \times \sharp\Delta_{\mathrm{iso}} = 8$, we have $\sharp\mathfrak{W} = 16$. One can arrange the elements of $\mathfrak{W}$ as follows:

	α_2	α	$-\alpha_2$	$-\alpha$
α_2	$(1, \mathbf{1}_{\alpha_2})$	$(r, \mathbf{1}_{\alpha_2})$	$(1, \mathbf{t}_{-\alpha_2})$	$(r, \mathbf{t}_{-\alpha_2})$
α	$(r, \mathbf{1}_{\alpha})$	$(1, \mathbf{1}_{\alpha})$	$(r, \mathbf{t}_{-\alpha})$	$(1, \mathbf{t}_{-\alpha})$
$-\alpha_2$	$(1, \mathbf{t}_{\alpha_2})$	$(r, \mathbf{t}_{\alpha_2})$	$(1, \mathbf{1}_{-\alpha_2})$	$(r, \mathbf{1}_{-\alpha_2})$
$-\alpha$	$(r, \mathbf{t}_{\alpha})$	$(1, \mathbf{t}_{\alpha})$	$(r, \mathbf{1}_{-\alpha})$	$(1, \mathbf{1}_{-\alpha})$

(2.9)

where each row and column respectively correspond to the codomain and domain of morphisms.

2.3. Definition of Coxeter groupoids

Coxeter groupoids were first introduced as semigroups in [3], and later on the notion is reformulated by Cuntz and Heckenberger [2] in terms of groupoids.

Let I be a finite index set, and Q a finitely generated free abelian group with base $\{\beta_i\}_{i \in I}$. Put $Q_+ := \sum_{i \in I} \mathbb{Z}_{\geq 0}\beta_i$. Let $C = (c_{i,j})_{i,j \in I}$ be a generalized Cartan matrix, i.e., a matrix with integer entries satisfying

(i) $c_{i,i} = 2$ and $c_{i,j} \leq 0$ $(i \neq j)$, and

(ii) if $c_{i,j} = 0$, then $c_{j,i} = 0$. Let us recall the definition of a Cartan scheme.

Definition 2.5. For a non-empty set A, the sets $\{\rho_i\}_{i\in I}$ of maps $\rho_i : A \to A$ and $\{C^a\}_{a\in A}$ of generalized Cartan matrices $C^a = (c^a_{i,j})$, the quadruple $\mathcal{C} := \mathcal{C}(I, A, \{\rho_i\}_{i\in I}, \{C^a\}_{a\in A})$ is called a Cartan scheme if it satisfies

(C1) $\rho_i^2 = \mathrm{id}_A$ $(\forall i \in I)$,

(C2) $c^{\rho_i(a)}_{i,j} = c^a_{i,j}$ $(\forall i, j \in I,\ \forall a \in A)$.

For $i \in I$ and $a \in A$, define $\sigma^a_i \in \mathrm{Aut}(Q)$ by

$$\sigma^a_i(\beta_j) = \beta_j - c^a_{i,j}\beta_i \quad (j \in I).$$

A root system associated with a Cartan scheme $\mathcal{C}$ is defined as follows:

Definition 2.6. Suppose that a subset $R^a \subset Q$ is given for each $a \in A$. Set $R^a_+ := R^a \cap Q_+$ and

$$m^a_{i,j} := \sharp\{R^a \cap (\mathbb{Z}_{\geq 0}\beta_i + \mathbb{Z}_{\geq 0}\beta_j)\} \quad (i, j \in I). \tag{2.10}$$

A pair $\mathcal{R} := \mathcal{R}(\mathcal{C}, \{R^a\}_{a\in A})$ of a Cartan scheme $\mathcal{C} = \mathcal{C}(I, A, \{\rho_i\}_{i\in I}, \{C^a\}_{a\in A})$ and the set $\{R^a\}_{a\in A}$ is called a root system of type $\mathcal{C}$ if it satisfies

(R1) $R^a = R^a_+ \cup (-R^a_+)$ $(\forall a \in A)$,

(R2) $R^a \cap \mathbb{Z}\beta_i = \{\beta_i, -\beta_i\}$ $(\forall i \in I, \forall a \in A)$,

(R3) $\sigma^a_i(R^a) = R^{\rho_i(a)}$ $(\forall i \in I, \forall a \in A)$,

(R4) if $m^a_{i,j} < \infty$ for $i, j \in I$ $(i \neq j)$ and $a \in A$, then $(\rho_i\rho_j)^{m^a_{i,j}}(a) = a$.

Let $\mathcal{R}$ be a root system of type $\mathcal{C}$.

Definition 2.7. The Coxeter groupoid $\mathcal{W}(\mathcal{R})$ associated with $\mathcal{R}$ is the groupoid defined by the following generators and relations.

(1) Objects: $\mathrm{Ob}(\mathcal{W}(\mathcal{R})) = A$,

(2) Generators: $\{s^a_i | i \in I,\ a \in A\}$, where $s^a_i \in \mathrm{Hom}_{\mathcal{W}(\mathcal{R})}(a, \rho_i(a))$,

(3) Relations:

$$s_i^2\mathbf{1}_a = \mathbf{1}_a, \quad (s_js_k)^{m^a_{j,k}}\mathbf{1}_a = \mathbf{1}_a \quad (\forall i, j, k \in I\ (j \neq k),\ \forall a \in A). \tag{2.11}$$

Here and after, we often abbreviate the superscript of generators determined automatically from $a \in A$ and use the following convention: For $i_1, \cdots, i_n \in I$, denote

$$s_{i_n} \cdots s_{i_2}s_{i_1}\mathbf{1}_a := s^{a_n}_{i_n} \circ \cdots \circ s^{a_2}_{i_2} \circ s^{a_1}_{i_1}, \tag{2.12}$$

where $a_1 := a$, $a_2 := \rho_{i_1}(a_1)$, $\cdots$, $a_n := \rho_{i_{n-1}}(a_{n-1})$.

Precisely speaking, in [2], the next definition is adopted.

Definition 2.8. The groupoid $\mathcal{W}^{\mathrm{CH}}(\mathcal{R})$ is defined by

(1) Objects: $\mathrm{Ob}(\mathcal{W}^{\mathrm{CH}}(\mathcal{R})) := A$,

(2) Morphisms: for a, $b \in A$, $\mathrm{Hom}_{\mathcal{W}^{\mathrm{CH}}(\mathcal{R})}(a, b)$ is the set of all triples (b, f, a) with $f = \sigma^{a_n}_{i_n} \circ \cdots \circ \sigma^{a_1}_{i_2} \circ \sigma^{a_1}_{i_1}$ and $b = \rho_{i_n}(a_n)$, where, for $i_1, \cdots, i_n \in I$, we put $a_1 := a$, $a_2 := \rho_{i_1}(a_1)$, $\cdots$, $a_n := \rho_{i_{n-1}}(a_{n-1})$.

(3) Composition: $(c, g, b) \circ (b, f, a) = (c, g \circ f, a)$.

In fact, two definitions are equivalent.

Theorem 2.1 ([3],[2])**.** *The Coxeter groupoid $\mathcal{W}(\mathcal{R})$ is isomorphic to the groupoid $\mathcal{W}^{\mathrm{CH}}(\mathcal{R})$, and an isomorphism is given by the functor which sends $s^a_i \mapsto (\rho_i(a), \sigma^a_i, a)$.*

For the details of proof, see [12].

2.4. Coxeter groupoids constructed from Weyl groups

Many Coxeter groupoids appeared in the representation theory of Lie superalgebras can be constructed as semi-direct product groupoids (Definition 2.3) from Weyl groups. For later use, we give typical examples of such Coxeter groupoids.

Let $\mathfrak{a}$ be a finite dimensional semi-simple Lie algebra of rank n. Set $I := \{1, 2, \cdots, n\}$. In this subsection, let $C = (c_{i,j})_{i,j\in I}$ be the Cartan matrix, Δ the root system and $\{\beta_i\}_{i\in I}$ the set of the simple roots of $\mathfrak{a}$. Let W be the Weyl group, whose generators (the simple reflections) are denoted by $\{s_i\}_{i\in I}$.

Suppose that W acts on a finite set A and regard A as a fine groupoid. We will show that the semi-direct product groupoid $W \ltimes A$ is a Coxeter groupoid.

Lemma 2.2. *Define the quadruple $\mathcal{C}^{W;A}$ and the pair $\mathcal{R}^{W;A}$ as follows:*

$$\mathcal{C}^{W;A} := \mathcal{C}(I, A, \{\rho_i\}_{i\in I}, \{C^a\}_{a\in A}), \quad \mathcal{R}^{W;A} := \mathcal{R}(\mathcal{C}^{W;A}, \{R^a\}_{a\in A}), \tag{2.13}$$

(1) *$\rho_i := s_i : A \to A$ $(i \in I)$ are given by the action of W on A,*

(2) *$C^a := C$ and $R^a := \Delta$ $(a \in A)$.*

Then, $\mathcal{C}^{W;A}$ is a Cartan scheme and $\mathcal{R}^{W;A}$ is a root system of type $\mathcal{C}^{W;A}$.

By the lemma, we obtain the Coxeter groupoid $\mathcal{W}(\mathcal{R}^{W;A})$ with $\mathcal{R}^{W;A}$. Let $\{s_i^a\}_{a\in A}$ be its generators, where $s_i^a \in \mathrm{Hom}_{\mathcal{W}(\mathcal{R}^{W;A})}(a, s_i(a))$.

Lemma 2.3. *The Coxeter groupoid $\mathcal{W}(\mathcal{R}^{W;A})$ is isomorphic to the semi-direct product groupoid $W \ltimes A$. An isomorphism is given by the functor which sends $s_i^a \mapsto [s_i, a]$, where the symbol $[s_i, a]$ is defined in (2.2). Hence, we have*

$$\sharp\mathcal{W}(\mathcal{R}^{W;A}) = \sharp W \times \sharp A. \tag{2.14}$$

Proof. Set $m_{i,j} := \sharp\{\Delta \cap (\mathbb{Z}_{\geq 0}\beta_i + \mathbb{Z}_{\geq 0}\beta_j)\}$. For any $a \in A$, the constant $m_{i,j}^a$ given by (2.10) is equal to $m_{i,j}$. Hence, the elements $\{[s_i, a] | i \in I, a \in A\}$ satisfy the relations (2.11), and there exists a homomorphism $\Phi : \mathcal{W}(\mathcal{R}^{W;A}) \to W \ltimes A$ of groupoid which sends $s_i^a \mapsto [s_i, a]$.

Since $\{[s_i, a] | i \in I, a \in A\}$ generate $W \ltimes A$, the homomorphism Φ is surjective. We will prove that it is injective. It suffices to show that for any

$$x = s_{i_p}^{a_p} \cdots s_{i_1}^{a_1} \in \mathcal{W}(\mathcal{R}^{W;A}) \quad (a_1 := a,\ a_{k+1} := s_{i_k}(a_k)\ (k = 1, \cdots, p)),$$

if $\Phi(x) = [1, a]$, then $x = \mathbf{1}_a$. Under the isomorphism in Theorem 2.1, x corresponds to the triple (b, w, a) with $w := s_{i_p} \cdots s_{i_1}$ and $b := a_{p+1}$. By the definition of Φ, we have $\Phi(x) = [w, a]$. Hence, if $\Phi(x) = [1, a]$, then $w = 1$ and $(b, w, a) = (a, 1, a)$ which corresponds to $\mathbf{1}_a$. Thus, we obtain $x = \mathbf{1}_a$, and the injectivity holds. □

To describe the groupoid part $\mathfrak{W}$ of a super Weyl groupoid, we will use $W \ltimes A$ constructed from the actions of W on A given below:

(1) Case $\mathfrak{a} = A_n$:

$$A := \{\eta_1, \eta_2, \cdots, \eta_{n+1}\}, \tag{2.15}$$

where $\{\eta_i\}_{i=1,\cdots,n+1}$ is the orthonormal base satisfying $\beta_i = \eta_i - \eta_{i+1}$.

(2) Case $\mathfrak{a} = B_n$, C_n or D_n:

$$A := \{\pm\eta_1, \pm\eta_2, \cdots, \pm\eta_n\}, \tag{2.16}$$

where $\{\eta_i\}_{i\in I}$ is orthogonal bases of $\mathfrak{h}^*$ satisfying

$$\beta_i = \begin{cases} \eta_i - \eta_{i+1} & (i < n) \\ \eta_n & (i = n \wedge \mathfrak{a} = B_n) \\ 2\eta_n & (i = n \wedge \mathfrak{a} = C_n) \\ \eta_{n-1} + \eta_n & (i = n \wedge \mathfrak{a} = D_n) \end{cases}.$$

3. Extensions of Coxeter groupoids and the groupoid part $\mathfrak{W}$

3.1. Involutive Coxeter groupoids and their extensions

Here, we introduce the notions of an involutive Coxeter groupoid and its extension.

Definition 3.1. Let $\mathcal{W}(\mathcal{R})$ be the Coxeter groupoid with a root system $\mathcal{R}$ of type $\mathcal{C}$, where $\mathcal{C} = \mathcal{C}(I, A, \{\rho_i\}_{i\in I}, \{C^a\}_{a\in A})$ and $\mathcal{R} = \mathcal{R}(\mathcal{C}, \{R^a\}_{a\in A})$. We say that $\mathcal{W}(\mathcal{R})$ is involutive if there exists a bijection

$$\bar{\cdot} : A \to A \tag{3.1}$$

with the following conditions:

(1) $\overline{(\bar{a})} = a$, $\bar{a} \neq a$ $(\forall a \in A)$,

(2) $\rho_i(\bar{a}) = \overline{\rho_i(a)}$ $(\forall i \in I, \forall a \in A)$,

(3) $C^{\bar{a}} = C^a$, $R^{\bar{a}} = R^a$ $(\forall a \in A)$.

In the cases of B_n, C_n D_n, the groupoid $\mathcal{W}(\mathcal{R}^{W;A})$ in (2.16) is an involutive Coxeter groupoid with the bijection $\bar{\cdot} : A \to A$ is given by $\bar{a} := -a$.

We introduce an extension $\mathcal{W}^e(\mathcal{R})$ of the involutive Coxeter groupoid $\mathcal{W}(\mathcal{R})$ with a root system $\mathcal{R}$ of type $\mathcal{C}$, where $\mathcal{R} = \mathcal{R}(\mathcal{C}, \{R^a\}_{a\in A})$ and $\mathcal{C} = \mathcal{C}(I, A, \{\rho_i\}_{i\in I}, \{C^a\}_{a\in A})$. Let $\bar{\cdot} : A \to A$ be the bijection (3.1).

Definition 3.2. Define the groupoid $\mathcal{W}^e(\mathcal{R})$ as follows:

(1) Objects: $\mathrm{Ob}(\mathcal{W}^e(\mathcal{R})) = A$,

(2) Generators: $\{s_i^a,\ t^a | i \in I,\ a \in A\}$, where $s_i^a \in \mathrm{Hom}_{\mathcal{W}^e(\mathcal{R})}(a, \rho_i(a))$, $t^a \in \mathrm{Hom}_{\mathcal{W}^e(\mathcal{R})}(a, \bar{a})$.

(3) Relations:

$$s_i^2 \mathbf{1}_a = \mathbf{1}_a, \quad (s_j s_k)^{m_{j,k}^a} \mathbf{1}_a = \mathbf{1}_a, \quad t^2 \mathbf{1}_a = \mathbf{1}_a, \quad s_i t \mathbf{1}_a = t s_i \mathbf{1}_a$$
$$(\forall i, j, k \in I\ (j \neq k),\ \forall a \in A), \tag{3.2}$$

where we use notation similar to (2.12) and abbreviate s_i^a and t^a to s_i and t respectively. For example, $s_i t \mathbf{1}_a = t s_i \mathbf{1}_a$ should be regarded as $s_i^{\bar{a}} t^a = t^{\rho_i(a)} s_i^a$.

Proposition 3.1. *The following equality holds:*

$$\mathcal{W}^e(\mathcal{R}) = \mathcal{W}(\mathcal{R}) \sqcup t\mathcal{W}(\mathcal{R}),$$

where the right hand side is the set-theoretic disjoint union. Hence, we have

$$\sharp \mathcal{W}^e(\mathcal{R}) = 2 \times \sharp \mathcal{W}(\mathcal{R}). \tag{3.3}$$

Proof. For simplicity, set $\mathcal{W} := \mathcal{W}(\mathcal{R})$ and let $\iota : \mathcal{W} \to \mathcal{W}$ be the isomorphism of groupoids given by

$$f = s_{i_n}^{a_n} \cdots s_{i_1}^{a_1} \mapsto \iota(f) := s_{i_n}^{\overline{a_n}} \cdots s_{i_1}^{\overline{a_1}}.$$

We will introduce appropriate groupoid structure on the set-theoretic disjoint union $\mathfrak{G} := \mathcal{W} \sqcup \mathcal{W}$ and show that it is isomorphic to $\mathcal{W}^e(\mathcal{R})$. Here, the groupoid $\mathfrak{G}$ is defined by

(1) Objects: $\mathrm{Ob}(\mathfrak{G}) := A$,

(2) Morphisms:

$$\mathrm{Hom}_{\mathfrak{G}}(a, b) := \mathrm{Hom}_{\mathcal{W}}(a, b) \sqcup \mathrm{Hom}_{\mathcal{W}}(a, \overline{b}),$$

where the right hand side is the set-theoretic disjoint union.

(3) Composition: for $f \in \mathrm{Hom}_{\mathfrak{G}}(a,b)$ and $g \in \mathrm{Hom}_{\mathfrak{G}}(b,c)$,

$$g \underset{\mathfrak{G}}{\circ} f := \begin{cases} g \circ f & (f \in \mathrm{Hom}_{\mathcal{W}}(a,b)) \\ \iota(g) \circ f & (f \in \mathrm{Hom}_{\mathcal{W}}(a,\overline{b})) \end{cases},$$

where to distinguish composition in $\mathfrak{G}$ and $\mathcal{W}$, we denote that of $\mathfrak{G}$ (resp. $\mathcal{W}$) by $\underset{\mathfrak{G}}{\circ}$ (resp. $\circ$).

Then, one can show that $\mathfrak{G}$ is a groupoid. Indeed, the inverse of $f \in \mathrm{Hom}_{\mathfrak{G}}(a,b)$ is given by

$$f^{-1} := \begin{cases} f^{-1} & (f \in \mathrm{Hom}_{\mathcal{W}}(a,b)) \\ \iota(f^{-1}) & (f \in \mathrm{Hom}_{\mathcal{W}}(a,\overline{b})) \end{cases}.$$

Moreover, if we define the functor $\Phi : \mathcal{W}^e(\mathcal{R}) \to \mathfrak{G}$ by

$$s_i^a \mapsto s_i^a, \quad t^a \mapsto \mathbf{1}_a \in \mathrm{Hom}_{\mathcal{W}}(a, \overline{(\bar{a})}) \subset \mathrm{Hom}_{\mathfrak{G}}(a, \bar{a}),$$

then Φ is an isomorphism. Hence, we obtain the proposition. □

3.2. Main results

Let $\mathfrak{W}$ be the groupoid part (2.5) associated with a basic classical Lie superalgebra $\mathfrak{g}$. Let $\mathfrak{g}_{\bar{0}}$ be the even part of $\mathfrak{g}$. In the following, $\mathfrak{g}'_{\bar{0}}$ denotes the semi-simple part of $\mathfrak{g}_{\bar{0}}$. Remark that $\Delta_{\bar{0}}$ is the root system of the semi-simple Lie algebra $\mathfrak{g}'_{\bar{0}}$. Let $\{\beta_i\}_{i \in I'}$ be the set of the simple roots of $\mathfrak{g}'_{\bar{0}}$ with an index set I'. For each $i \in I'$, we denote the real reflection corresponding to β_i by s_i.

Let $\mathcal{W}(\mathcal{R}^{W;\Delta_{\mathrm{iso}}})$ be the Coxeter groupoid given by Lemma 2.2 for $\mathfrak{a} := \mathfrak{g}'_{\bar{0}}$. By Lemma 2.3, we have

Lemma 3.1. *The semi-direct product $W \ltimes \Delta_{\mathrm{iso}}$ is isomorphic to $\mathcal{W}(\mathcal{R}^{W;\Delta_{\mathrm{iso}}})$, and an isomorphism is given by the functor $\mathcal{W}(\mathcal{R}^{W;\Delta_{\mathrm{iso}}}) \to W \ltimes \Delta_{\mathrm{iso}}$ which sends*

$$s_i^{\gamma} \longmapsto (s_i, \mathbf{1}_{s_i(\gamma)}) \quad (i \in I',\ \gamma \in \Delta_{\mathrm{iso}}). \tag{3.4}$$

Moreover, if we define the bijection $\bar{\cdot} : \Delta_{\mathrm{iso}} \to \Delta_{\mathrm{iso}}$ by $\bar{\gamma} := -\gamma$, then it is an involutive Coxeter groupoid.

As a corollary, we obtain the following theorem:

Theorem 3.1. *The groupoid part $\mathfrak{W}$ of the super Weyl groupoid is isomorphic to the extension $\mathcal{W}^e(\mathcal{R}^{W;\Delta_{\rm iso}})$ of the involutive Coxeter groupoid $\mathcal{W}(\mathcal{R}^{W;\Delta_{\rm iso}})$.*

Proof. One can extend (3.4) to a homomorphism $\mathcal{W}^e(\mathcal{R}^{W;\Delta_{\rm iso}}) \to \mathfrak{W}$ by $t^\gamma \mapsto (1, \mathbf{t}_\gamma)$ $(\gamma \in \Delta_{\rm iso})$. It is surjective since $\mathfrak{W}$ is generated by $\{s_i^\gamma, t^\gamma | i \in I', \gamma \in \Delta_{\rm iso}\}$. By (2.7) and (3.3), we have $\sharp\mathfrak{W} = \sharp\mathcal{W}^e(\mathcal{R}^{W;\Delta_{\rm iso}})$. Hence, it is also injective. We complete the proof. □

4. Description of the groupoid part $\mathfrak{W}$

The Weyl group of a basic classical Lie superalgebra $\mathfrak{g}$ coincides with that of the semi-simple part $\mathfrak{g}_{\bar{0}}'$. From this fact, we obtain a description of $\mathfrak{W}$ in terms of the Weyl groups of simple Lie algebras. Here, we consider the case where $\mathfrak{g}$ is of types $A(m,n)$, $B(m,n)$, $C(n)$ and $D(m,n)$.

4.1. Preliminaries

In this subsection, we will show that the direct product (2.4) is a Coxeter groupoid if $\mathfrak{G}_1$ and $\mathfrak{G}_2$ are Coxeter groupoids. We first introduce the product of two Cartan schemes.

Lemma 4.1. *For Cartan schemes $\mathcal{C}_\ell := \mathcal{C}(I_\ell, A_\ell, \{\rho_{\ell;i}\}_{i\in I_\ell}, \{C_\ell^a\}_{a\in A_\ell})$ and root systems $\mathcal{R}_\ell := \mathcal{R}(\mathcal{C}_\ell, \{R_\ell^a\})$ of $\mathcal{C}_\ell$ with $\ell = 1, 2$, define the quadruple $\mathcal{C}$ and the pair $\mathcal{R}$ by*

$$\mathcal{C} := \mathcal{C}(I, A, \{\rho_i\}_{i\in I}, \{C^a\}_{a\in A}), \quad \mathcal{R} := \mathcal{R}(\mathcal{C}, \{R^a\}_{a\in A}), \tag{4.1}$$

(1) *$I = I_1 \sqcup I_2$,*

(2) *$A := A_1 \times A_2$,*

(3) *For $i \in I$, the maps $\rho_i : A \to A$ are given by*

$$\rho_i((a,b)) := \begin{cases} (\rho_{1;i}(a), b) & (i \in I_1) \\ (a, \rho_{2;i}(b)) & (i \in I_2) \end{cases}.$$

(4) *For $(a,b) \in A$,*

$$C^{(a,b)} := \left(\begin{array}{c|c} C_1^a & 0 \\ \hline 0 & C_2^b \end{array}\right) \quad \text{and} \quad R^{(a,b)} := R^a \sqcup R^b.$$

Then, $\mathcal{C}$ is a Cartan scheme, and $\mathcal{R}$ is a root system of type $\mathcal{C}$.

We denote the Cartan scheme $\mathcal{C}$ and the root system $\mathcal{R}$ in the lemma by $\mathcal{C}_1 \times \mathcal{C}_2$ and $\mathcal{R}(\mathcal{C}_1 \times \mathcal{C}_2)$. Then, one can define the Coxeter groupoid $\mathcal{W}(\mathcal{R}(\mathcal{C}_1 \times \mathcal{C}_2))$ with $\mathcal{R}(\mathcal{C}_1 \times \mathcal{C}_2)$. Let $\{s_i^a | i \in I_\ell,\ a \in A_\ell\}$ and $\{s_i^{(a,b)} | i \in I_1 \sqcup I_1,\ a \in A_1,\ b \in A_2\}$ be the generators of $\mathcal{W}(\mathcal{R}_\ell)$ and $\mathcal{W}(\mathcal{R}(\mathcal{C}_1 \times \mathcal{C}_2))$. Let $\mathcal{W}(\mathcal{R}_1) \times \mathcal{W}(\mathcal{R}_2)$ be the direct product (2.4) of $\mathcal{W}(\mathcal{R}_1)$ and $\mathcal{W}(\mathcal{R}_2)$.

Proposition 4.1. *The Coxeter groupoid $\mathcal{W}(\mathcal{R}(\mathcal{C}_1 \times \mathcal{C}_2))$ is isomorphic to $\mathcal{W}(\mathcal{R}_1) \times \mathcal{W}(\mathcal{R}_2)$, and an isomorphism is given by the functor ϕ : $\mathcal{W}(\mathcal{R}(\mathcal{C}_1 \times \mathcal{C}_2)) \to \mathcal{W}(\mathcal{R}_1) \times \mathcal{W}(\mathcal{R}_2)$ which sends*

$$s_i^{(a,b)} \mapsto \begin{cases} (s_i^a, \mathbf{1}_b) & (i \in I_1) \\ (\mathbf{1}_a, s_i^b) & (i \in I_2) \end{cases}.$$

Hence, we have

$$\sharp\mathcal{W}(\mathcal{R}(\mathcal{C}_1 \times \mathcal{C}_2)) = \sharp\mathcal{W}(\mathcal{R}_1) \times \sharp\mathcal{W}(\mathcal{R}_2). \tag{4.2}$$

Next, we consider an operation to provide an involutive Coxeter groupoid. Let $\mathcal{W}(\mathcal{R})$ be a not necessarily involutive Coxeter groupoid with the root system $\mathcal{R}$ of type $\mathcal{C}$, where $\mathcal{C} = \mathcal{C}(I, A, \{\rho_i\}_{i \in I}, \{C^a\}_{a \in A})$ and $\mathcal{R} = \mathcal{R}(\mathcal{C}, \{R^a\}_{a \in A})$.

For the set A in the quadruple $\mathcal{C}$, we set $\breve{A} := A_+ \sqcup A_-$, where $A_\pm$ are copies of A, and denote $a \in A_\pm$ by $\pm a$. Define the bijection $\bar{\cdot} : \breve{A} \to \breve{A}$ by $\overline{(\pm a)} := \mp a$. Then, we have

Lemma 4.2. *For the above $\mathcal{C}$ and $\mathcal{R}$, define the Cartan scheme $\breve{\mathcal{C}}$ and the root system $\breve{\mathcal{R}}$ by*

$$\breve{\mathcal{C}} := \mathcal{C}(I, \breve{A}, \{\breve{\rho}_i\}_{i \in I}, \{\breve{C}^a\}_{a \in \breve{A}}), \quad \breve{\mathcal{R}} := \mathcal{R}(\breve{\mathcal{C}}, \{\breve{R}^a\}_{a \in A}), \tag{4.3}$$

(1) $\breve{\rho}_i(\pm a) := \pm \rho_i(a)$ $(i \in I,\ a \in A)$,

(2) $\breve{C}^{\pm a} := C^a$ *and* $\breve{R}^{\pm a} := R^a$ $(a \in A)$.

Then, the Coxeter groupoid $\mathcal{W}(\breve{\mathcal{R}})$ is involutive.

Let $\mathcal{W}(\mathcal{R}) \sqcup \mathcal{W}(\mathcal{R})$ be the disjoint union (2.3) of the same Coxeter groupoids $\mathcal{W}(\mathcal{R})$. One can easily show the following proposition:

Proposition 4.2. *The Coxeter groupoid $\mathcal{W}(\breve{\mathcal{R}})$ is isomorphic to $\mathcal{W}(\mathcal{R}) \sqcup \mathcal{W}(\mathcal{R})$, and an isomorphism is given by the functor which sends*

$$s_i^{+a} \mapsto (s_i^a)_1, \quad s_i^{-a} \mapsto (s_i^a)_2,$$

where the subscript ℓ in $(s_i^a)_\ell$ indicates that s_i^a is in the first ($\ell = 1$) (resp. the second ($\ell = 2$)) component of the disjoint union. Hence, we have

$$\sharp\mathcal{W}(\breve{\mathcal{R}}) = 2 \times \sharp\mathcal{W}(\mathcal{R}). \tag{4.4}$$

Finally, we give three examples: (i) $\mathcal{W}(\mathcal{R}^{W;A})$ with $\mathcal{R}^{W;A}$ given in (2.13), (ii) $\mathcal{W}(\breve{\mathcal{R}}^{W;A})$ with $\breve{\mathcal{R}}^{W;A}$ given in Lemma 4.2, (iii) the extension $\mathcal{W}^e(\breve{\mathcal{R}}^{W;A})$, in the case where $\mathfrak{a} = \mathfrak{sl}_2$.

Example 4.1. Suppose that $\mathfrak{a} = \mathfrak{sl}_2$ (of type A_1). Then, the Weyl group is given by $W = \{1, s\}$, where s is a unique simple reflection. The action of W on the set $A = \{\eta_1, \eta_2\}$ in (2.15) is given by $s(\eta_i) = \eta_{3-i}$ $(i = 1, 2)$. The fact that $\sharp\mathcal{W}(\mathcal{R}^{W;A}) = \sharp W \times \sharp A = 4$ implies $\sharp\mathcal{W}(\breve{\mathcal{R}}^{W;A}) = 8$ and $\sharp\mathcal{W}^e(\breve{\mathcal{R}}^{W;A}) = 16$. We arrange the elements of these groupoids in a way similar to (2.9):

(1) $\mathcal{W}(\mathcal{R}^{W;A})$:

	η_1	η_2
η_1	$\mathbf{1}_{\eta_1}$	s^{η_2}
η_2	s^{η_1}	$\mathbf{1}_{\eta_2}$

(4.5)

(2) $\mathcal{W}(\breve{\mathcal{R}}^{W;A})$:

	η_1	η_2	$-\eta_1$	$-\eta_2$
η_1	$\mathbf{1}_{\eta_1}$	s^{η_2}		
η_2	s^{η_1}	$\mathbf{1}_{\eta_2}$		
$-\eta_1$			$\mathbf{1}_{-\eta_1}$	$s^{-\eta_2}$
$-\eta_2$			$s^{-\eta_1}$	$\mathbf{1}_{-\eta_2}$

(4.6)

where the blanks mean the corresponding Hom-sets are empty.

(3) $\mathcal{W}^e(\breve{\mathcal{R}}^{W;A})$:

	η_1	η_2	$-\eta_1$	$-\eta_2$
η_1	$\mathbf{1}_{\eta_1}$	s^{η_2}	$t^{-\eta_1}$	$t^{-\eta_1}s^{-\eta_2}$
η_2	s^{η_1}	$\mathbf{1}_{\eta_2}$	$t^{-\eta_2}s^{-\eta_1}$	$t^{-\eta_2}$
$-\eta_1$	t^{η_1}	$t^{\eta_1}s^{\eta_2}$	$\mathbf{1}_{-\eta_1}$	$s^{-\eta_2}$
$-\eta_2$	$t^{\eta_2}s^{\eta_1}$	t^{η_2}	$s^{-\eta_1}$	$\mathbf{1}_{-\eta_2}$

(4.7)

Remark that the above $\mathcal{W}(\breve{\mathcal{R}}^{W;A})$ is isomorphic to the groupoid $\mathfrak{W}$ given in Example 2.1 for $\mathfrak{g} = \mathfrak{sl}(2,1)$. In the following subsection, we consider similar isomorphisms for general basic classical Lie superalgebras.

4.2. Explicit description

For the basic classical Lie superalgebra $\mathfrak{g}$ of type $X(m,n)$, we denote the groupoid part $\mathfrak{W}$ defined in (2.5) by $\mathfrak{W}^{X(m,n)}$. For the simple Lie algebra $\mathfrak{a}$ of type X_n, we also denote the Cartan scheme $\mathcal{C}^{W;A}$ and the root system $\mathcal{R}^{W;\Lambda}$ given in (2.13), (2.15) and (2.16) by $\mathcal{C}^{X_m}$ and $\mathcal{R}^{X_m}$ respectively.

Theorem 4.1. *The groupoid part $\mathfrak{W}^{X(m,n)}$ is isomorphic to the extension of the following involutive Coxeter groupoid:*

(1) *Case $\mathfrak{g} = A(m,n)$: $\mathfrak{W}^{A(m,n)} \simeq \mathcal{W}^e(\breve{\mathcal{R}}(\mathcal{C}^{A_m} \times \mathcal{C}^{A_n}))$,*

(2) *Case $\mathfrak{g} = B(m,n)$: $\mathfrak{W}^{B(m,n)} \simeq \mathcal{W}^e(\mathcal{R}(\mathcal{C}^{B_m} \times \mathcal{C}^{C_n}))$,*

(3) *Case $\mathfrak{g} = C(m,n)$: $\mathfrak{W}^{C(n)} \simeq \mathcal{W}^e(\breve{\mathcal{R}}(\mathcal{C}^{C_{n-1}}))$,*

(4) *Case $\mathfrak{g} = D(m,n)$: $\mathfrak{W}^{D(m,n)} \simeq \mathcal{W}^e(\mathcal{R}(\mathcal{C}^{D_m} \times \mathcal{C}^{C_n}))$.*

In the case of $\mathfrak{g} = A(m,0)$, $\breve{\mathcal{R}}(\mathcal{C}^{A_m} \times \mathcal{C}^{A_n})$ should be regarded as $\breve{\mathcal{R}}(\mathcal{C}^{A_m})$.

Proof. At first, we will prove the theorem in the case where $\mathfrak{g} = A(m,n)$. By Lemma 3.1 and Theorem 3.1, we may prove that there exists an isomorphism

$$\Phi : \mathcal{W}(\breve{\mathcal{R}}(\mathcal{C}^{A_m} \times \mathcal{C}^{A_n})) \longrightarrow W^{A(m,n)} \ltimes \Delta_{\mathrm{iso}}, \tag{4.8}$$

where $W^{A(m,n)}$ denotes the Weyl group of $\mathfrak{g}$.

Let $\mathfrak{a}$ be the simple Lie algebra of type A_n and W^{A_n} its Weyl group. We consider $I_{m+n+1} := \{1, 2, \cdots, m+n+1\}$ and $I_n := \{1, 2, \cdots, n\}$ as the index sets of the simple roots of $\mathfrak{g}$ and $\mathfrak{a}$ respectively. To distinguish notation for $\mathfrak{g}$ (resp. $\mathfrak{a}$), the kth simple root by α_k (resp. β_k) and the kth simple reflection by r_k (resp. s_k). Here, we use the following data on the root system of $\mathfrak{g}$.

(1)
$$\alpha_k = \begin{cases} \epsilon_k - \epsilon_{k+1} & (1 \le k \le m) \\ \epsilon_{m+1} - \delta_1 & (k = m+1) \\ \delta_{k-m-1} - \delta_{k-m} & (m+2 \le k \le m+n+1) \end{cases},$$

where $\{\epsilon_i, \delta_j | 1 \le i \le m+1,\ 1 \le j \le n+1\}$ is an orthogonal base satisfying $(\epsilon_i, \epsilon_i) = 1$ and $(\delta_j, \delta_j) = -1$.

(2) $\Delta_{\mathrm{iso}} = \{\pm(\epsilon_i - \delta_j) | 1 \le i \le m+1,\ 1 \le j \le n+1\}$,

(3) $W^{A(m,n)}$ is generated by $\{r_k | k \in I_{m+n+1} \setminus \{m+1\}\}$, and

$$W^{A(m,n)} \simeq W^{A_m} \times W^{A_n}. \tag{4.9}$$

(4) $\Delta_{\mathrm{iso}} = \Delta_{\bar{1}}^+ \sqcup (-\Delta_{\bar{1}}^+)$: the orbits with respect to the action of $W^{A(m,n)}$.

Recall that W^{A_n} acts on the set $\{\eta_1, \cdots, \eta_{n+1}\}$ as in (2.15). Put

$$I_{m,n} := I_m \sqcup I_n, \quad A_{m,n} := \{\eta_1, \cdots, \eta_{m+1}\} \times \{\eta_1, \cdots, \eta_{n+1}\}.$$

Let us consider a natural action of $W^{A_m} \times W^{A_n}$ on the set $A_{m,n}$. By abuse of notation, for each $k \in I_{m,n}$, we define the reflection $s_k \in W^{A_m} \times W^{A_n}$ by

$$s_k := \begin{cases} (s_k, 1_n) & (k \in I_m) \\ (1_m, s_k) & (k \in I_n) \end{cases},$$

where 1_n denotes the unit of W^{A_n}.

Let $\phi : W^{A_m} \times W^{A_n} \to W^{A(m,n)}$ be the inverse of the isomorphism (4.9). We may suppose that ϕ is given by

$$\phi(s_k) = \begin{cases} r_k & (k \in I_m) \\ r_{m+1+k} & (k \in I_n) \end{cases}.$$

The following lemma is a key of our proof.

Lemma 4.3. *Define the bijection $\phi : A_{m,n} \to \Delta_{\mathrm{iso}}^+$ by*

$$\phi((\eta_i, \eta_j)) := \epsilon_i - \delta_j. \tag{4.10}$$

Then, ϕ commutes with the action of $W^{A_m} \times W^{A_n}$, namely,

$$\phi(x.(\eta_i, \eta_j)) = \phi(x).\phi((\eta_i, \eta_j))$$

holds for any $x \in W^{A_m} \times W^{A_n}$ and $(\eta_i, \eta_j) \in A_{m,n}$.

By means of the correspondence (4.10), we define the isomorphism Φ in (4.8) by

$$\Phi(s_k^{\pm(\eta_i, \eta_j)}) := [\phi(s_k), \pm(\epsilon_i - \delta_j)].$$

In fact, one can directly check that Φ is a homomorphism of groupoid. Since $\{[\phi(s_k), \pm(\epsilon_i - \delta_j)] | k \in I_{m,n}, (\epsilon_i - \delta_j) \in A_{m,n}$ generate $W \ltimes \Delta_{\mathrm{iso}}$, Φ is surjective. Moreover, (2.14), (4.2) and (4.4) imply

$$\sharp \mathcal{W}(\breve{\mathcal{R}}(\mathcal{C}^{A_m} \times \mathcal{C}^{A_n})) = 2 \times (m+1)!(m+1) \times (n+1)!(n+1).$$

The right hand side equals to $\sharp(W^{A(m,n)} \ltimes \Delta_{\mathrm{iso}})$, since $\sharp\Delta_{\mathrm{iso}} = 2(m+1)(n+1)$. Hence, Φ is bijective. We have proved the theorem for $A(m,n)$.

Next, we consider the case of $B(m,n)$. It is enough to show that there exists an isomorphism

$$\Phi : \mathcal{W}(\mathcal{R}(\mathcal{C}^{B_m} \times \mathcal{C}^{C_n})) \longrightarrow W^{B(m,n)} \ltimes \Delta_{\mathrm{iso}}, \tag{4.11}$$

where $W^{B(m,n)}$ denotes the Weyl group of $B(m,n)$.

Let W^{B_m} and W^{C_n} be the Weyl groups of the Lie algebras of type B_m and C_n respectively. Let us recall the following description on the simple roots and the Weyl group of $B(m,n)$.

(1)

$$\alpha_k = \begin{cases} \delta_k - \delta_{k+1} & (1 \le k \le n-1) \\ \delta_n - \epsilon_1 & (k = n) \\ \epsilon_{k-n} - \epsilon_{k-n+1} & (n+1 \le k \le m+n-1) \\ \epsilon_m & (k = m+n) \end{cases},$$

where $\{\epsilon_i, \delta_j | 1 \le i \le m,\ 1 \le j \le n\}$ is an orthogonal base satisfying $(\delta_j, \delta_j) = 1$ and $(\epsilon_i, \epsilon_i) = -1$.

(2) $\Delta_{\mathrm{iso}} = \{\sigma_1\epsilon_i + \sigma_2\delta_j | i \in I_m, j \in I_n, \sigma_1, \sigma_2 \in \{\pm\}\}$,

(3) $W^{B(m,n)}$ is generated by the real reflections $\{r_i | k \ne n\} \cup \{r_{2\delta_n}\}$, and

$$W^{B(m,n)} \simeq W^{B_m} \times W^{C_n}. \tag{4.12}$$

(4) $W^{B(m,n)}$ acts on Δ_{iso} transitively.

Let $\{s_k\}_{k \in I_m}$ and $\{s_k\}_{k \in I_n}$ be the generators of the Coxeter groups W^{B_m} and W^{C_n}. They act on the sets $\{\pm\eta_i\}_{i \in I_m}$ and $\{\pm\eta_j\}_{j \in I_n}$ given in (2.16). Put

$$I_{m,n} := I_m \sqcup I_n, \quad A_{m,n} := \{\pm\eta_1, \cdots, \pm\eta_m\} \times \{\pm\eta_1, \cdots, \pm\eta_n\}.$$

Let $\phi : W^{B(m,n)} \to W^{B_m} \times W^{C_n}$ be the inverse of the isomorphism (4.12). Unlike the case of $A(m,n)$, $W^{B(m,n)}$ transitively acts on Δ_{iso}, and the following lemma holds:

Lemma 4.4. *Define the bijection* $\phi : A_{m,n} \to \Delta_{\mathrm{iso}}$ *by*

$$\phi((\sigma_1\eta_i, \sigma_2\eta_j)) := \sigma_1\epsilon_i + \sigma_2\delta_j \quad (\sigma_1, \sigma_2 \in \{\pm\})$$

Then, ϕ *commutes with the action of* $W^{B_m} \times W^{C_n}$.

One can prove that Φ is an isomorphism by arguments similar to the case of $A(m,n)$. Now, we have proved the theorem for $B(m,n)$.

In the cases where $\mathfrak{g} = C(n)$ (resp. $D(m,n)$), the theorem can be shown in ways similar to $A(m,n)$ (resp. $B(m,n)$), since Δ_{iso} consists of two orbits $\pm\Delta_{\mathrm{iso}}^{+}$ (resp. one orbit Δ_{iso}) with respect to the action of the Weyl group. □

Remark 4.1. The structure of $\mathfrak{W}^{X(m,n)}$ is related to the Weyl group $W^{X(m,n)}$ and the orbits of Δ_{iso} with respect to the action of $W^{X(m,n)}$.

$X(m,n)$	$W^{X(m,n)}$	Orbits
$A(m,n)$	$W^{A_m} \times W^{A_n}$	$\pm\Delta_{\mathrm{iso}}^{+}$
$B(m,n)$	$W^{B_m} \times W^{C_n}$	Δ_{iso}
$C(n)$	$W^{C_{n-1}}$	$\pm\Delta_{\mathrm{iso}}^{+}$
$D(m,n)$	$W^{D_m} \times W^{C_n}$	Δ_{iso}

5. Generalized Verma modules and the groupoid $\mathfrak{W}$

Here, we discuss a relation between certain generalized Verma modules over $\mathfrak{sl}(2,1)$ with atypical highest weight and the groupoid $\mathfrak{W}^{A(1,0)}$. For basics in the representation theory of Lie superalgebras needed in this and the next sections, the reader can consult [8] and [10].

5.1. Definition of generalized Verma modules

For simplicity, we set $\mathfrak{g} := \mathfrak{sl}(2,1)$. Let $W = \{1, r\}$ be the Weyl group of $\mathfrak{g}$, where r denotes a (unique) even reflection. Let Σ be one of the bases Π_k ($k = 0,1,2$) of $\mathfrak{g}$ given in (A.1). We define the triangular decomposition of $\mathfrak{g}$ associated with Σ as follows:

$$\mathfrak{g} = \mathfrak{g}_{\Sigma}^{+} \oplus \mathfrak{h} \oplus \mathfrak{g}_{\Sigma}^{-}, \quad \mathfrak{g}_{\Sigma}^{\pm} := \bigoplus_{\gamma \in \Delta_{\Sigma}^{\pm}} \mathfrak{g}^{\gamma},$$

where $\Delta_{\Sigma}^{\pm} := \Delta \cap Q_{\Sigma}^{\pm}$ ($Q_{\Sigma}^{\pm} := \pm\sum_{\gamma\in\Sigma} \mathbb{Z}_{\geq 0}\gamma$), and $\mathfrak{g}^{\gamma}$ denotes a root subspace of $\mathfrak{g}$. Let $\rho_{\Sigma} \in \mathfrak{h}^*$ be the Weyl vector associated with Σ, which satisfies $(\rho_{\Sigma}, \gamma) = \frac{1}{2}(\gamma,\gamma)$ ($\gamma \in \Sigma$).

For later use, we introduce notation. Let $\mathfrak{b}_{\Sigma} := \mathfrak{g}_{\Sigma}^{+} \oplus \mathfrak{h}$ be the Borel subalgebra associated with Σ and $M_{\Sigma}(\Lambda)$ the Verma module with highest weight $\Lambda \in \mathfrak{h}^*$. We denote a highest weight vector of $M_{\Sigma}(\Lambda)$ by $\mathbf{1}_{\Sigma}^{\Lambda}$. Let $L_{\Sigma}(\Lambda)$ be the irreducible quotient of $M_{\Sigma}(\Lambda)$.

We are interested in the structure of the generalized Verma modules defined below. For Σ and an isotropic odd root $\tau \in \Sigma$, define a parabolic subalgebra by $\mathfrak{p}_{\Sigma;\tau} := \mathfrak{b}_\Sigma \oplus \mathbb{C}f_\tau$, where $f_\tau \in \mathfrak{g}^{-\tau}$ denotes a root vector. For $\Lambda \in \mathfrak{h}^*$ such that $(\Lambda, \tau) = 0$, let $\mathbb{C}\mathbf{1}^\Lambda_{\Sigma;\tau}$ be the 1-dimensional $\mathfrak{p}_{\Sigma;\tau}$-module defined by $\mathfrak{g}^+_\Sigma \mathbf{1}^\Lambda_{\Sigma;\tau} = \{0\}$, $h.\mathbf{1}^\Lambda_{\Sigma;\tau} = \Lambda(h)\mathbf{1}^\Lambda_{\Sigma;\tau}$ $(h \in \mathfrak{h})$ and $f_\tau \mathbf{1}^\Lambda_{\Sigma;\tau} = 0$. Let $U(\mathfrak{g})$ be the enveloping algebra of $\mathfrak{g}$. We define a generalized Verma module $N_\Sigma(\Lambda; \tau)$ as follows:

Definition 5.1. $N_\Sigma(\Lambda; \tau) := U(\mathfrak{g}) \otimes_{U(\mathfrak{p}_{\Sigma;\tau})} \mathbb{C}\mathbf{1}^\Lambda_{\Sigma;\tau}$.

It should be noted that $f_\tau \mathbf{1}^\Lambda_\Sigma$ is a singular vector of $M_\Sigma(\Lambda)$ and

$$N_\Sigma(\Lambda; \tau) \simeq M_\Sigma(\Lambda)/U(\mathfrak{g})f_\tau \mathbf{1}^\Lambda_\Sigma. \tag{5.1}$$

Let r_τ be the odd reflection defined by τ, and set $\Sigma' := r_\tau(\Sigma)$.

Proposition 5.1. *There exists an isomorphism of* $\mathfrak{g}$*-modules:*

$$N_\Sigma(\Lambda; \tau) \simeq N_{\Sigma'}(\Lambda; -\tau). \tag{5.2}$$

Proof. Since $\Sigma \cup \{\tau\} = \Sigma' \cup \{-\tau\}$ by Lemma 3.3 in [10], we have $\mathfrak{p}_{\Sigma;\tau} = \mathfrak{p}_{\Sigma';-\tau}$ and hence, there exists the isomorphism which sends $\mathbf{1}^\Lambda_{\Sigma;\tau}$ to $\mathbf{1}^\Lambda_{\Sigma';-\tau}$. □

For other fundamental properties of $N_\Sigma(\Lambda; \tau)$, see [10].

5.2. Action of the groupoid $\mathfrak{W}$

In [9], Sergeev and Veselov introduced an action of the super Weyl groupoid on $\mathfrak{h}^*$, and show that the representation rings for basic classical Lie superalgebras can be regarded as invariants with respect to the action. Here we consider the restriction of their action to the groupoid part $\mathfrak{W}$.

For $\alpha \in \Delta_{\mathrm{iso}}$, define the subspace $\mathfrak{h}^*_\alpha$ of $\mathfrak{h}^*$ by

$$\mathfrak{h}^*_\alpha := \{\lambda \in \mathfrak{h}^* | (\lambda, \alpha) = 0\}, \tag{5.3}$$

and consider the union $\bigcup_{\alpha \in \Delta_{\mathrm{iso}}} \mathfrak{h}^*_\alpha \subset \mathfrak{h}^*$. Remark that $\mathfrak{h}^*_\alpha = \mathfrak{h}^*_{-\alpha}$. Let us introduce the action of $\mathfrak{W}$ on the union as follows: For each $\alpha \in \Delta_{\mathrm{iso}}$, we define the maps $(w, \mathbf{1}_{w(\alpha)})$, $(w, \mathbf{t}_{w(\alpha)}) : \mathfrak{h}^*_\alpha \to \mathfrak{h}^*_{w(\alpha)}$ by

$$(w, \mathbf{1}_{w(\alpha)})(\lambda) := w(\lambda), \quad (w, \mathbf{t}_{w(\alpha)})(\lambda) := w(\lambda + \alpha), \tag{5.4}$$

where $\lambda \in \mathfrak{h}^*_\alpha$. Then, we have

Lemma 5.1. *The groupoid* $\mathfrak{W}$ *acts on the union* $\bigcup_{\alpha \in \Delta_{\mathrm{iso}}} \mathfrak{h}^*_\alpha$ *by (5.4).*

5.3. Embedding diagrams of $N_\Sigma(\Lambda;\tau)$

For $\nu \in \Delta_{\rm iso}$, let us consider

$$\{x \in \mathfrak{W}^{A(1,0)} | \mathrm{dom}(x) = \nu\} = \{(1, \mathbf{1}_\nu), (1, \mathbf{t}_\nu)(r, \mathbf{1}_{r(\nu)}), (r, \mathbf{t}_{r(\nu)})\},$$

where $\mathrm{dom}(x)$ denotes the domain of $x \in \mathfrak{W}^{A(1,0)}$. It is a column of the super Weyl groupoid $\mathfrak{W}^{A(1,0)}$ in Example 2.1. Here, we will show that each column is related to the embeddings of generalized Verma modules given in Theorem A.1 and Remark A.1.

For any $\sigma \in \Delta_{\rm iso}$, there uniquely exists a base Σ containing σ. We denote it by $\Sigma[\sigma]$. For $x \in \mathrm{Hom}_{\mathfrak{W}}(\sigma, \tau)$ and an atypical integrable highest weight $\Lambda \in \mathfrak{h}^*_\sigma$ (see (A.3)), set

$$N[x] := N_{\Sigma[\tau]}(x(\Lambda + \rho_{\Sigma[\sigma]}) - \rho_{\Sigma[\tau]}; \tau),$$

where $x(\Lambda + \rho_{\Sigma[\sigma]})$ is given by the action (5.4). Then, we have

Proposition 5.2. *There exists following diagram of generalized Verma modules:*

$$\begin{array}{ccc} N[(1,\mathbf{1}_\nu)] & \overset{\sim}{\text{———}} & N[(1,\mathbf{t}_\nu)] \\ \uparrow & & \uparrow \\ N[(r,\mathbf{1}_{r(\nu)})] & \overset{\sim}{\text{———}} & N[(r,\mathbf{t}_{r(\nu)})] \end{array}$$

Proof. We use the notations in Section A. Since $\Delta_{\rm iso} = \{\alpha_2, \beta_1, \beta_2, \gamma_1\}$, the vertical arrows correspond to the embeddings given in Theorem A.1 and Remark A.1. Let us look at the horizontal isomorphisms. For each $w \in W = \{1, r\}$, we have

$$N[(w, \mathbf{1}_{w(\nu)})] = N_{\Pi[w(\nu)]}(w(\Lambda + \rho_{\Pi[\nu]}) - \rho_{\Pi[w(\nu)]}; w(\nu)),$$
$$N[(w, \mathbf{t}_{w(\nu)})] = N_{\Pi[-w(\nu)]}(w(\Lambda + \rho_{\Pi[\nu]} + \nu) - \rho_{\Pi[-w(\nu)]}; -w(\nu)).$$

Since $\Pi[-w(\nu)] = r_{w(\nu)}(\Pi[w(\nu)])$ and $\rho_{\Pi[-w(\nu)]} = \rho_{\Pi[w(\nu)]} + w(\nu)$, we have

$$w(\Lambda + \rho_{\Pi[\nu]}) - \rho_{\Pi[w(\nu)]} = w(\Lambda + \rho_{\Pi[\nu]} + \nu) - \rho_{\Pi[-w(\nu)]}.$$

Hence, $N[(w, \mathbf{1}_{w(\nu)})] \simeq N[(w, \mathbf{t}_{w(\nu)})]$ by Proposition 5.1. $\square$

A. Structure of $N_\Sigma(\Lambda;\tau)$ with atypical highest weight

In this appendix, we will describe the structure of generalized Verma modules $N_\Sigma(\Lambda;\tau)$ whose irreducible quotients are atypical integrable representations. For ordinary Verma modules over $\mathfrak{sl}(2,1)$, the complete description of the structure is provided in [7].

In the case of $\mathfrak{sl}(2,1)$, there are three conjugacy class of bases with respect to the action of the Weyl group W and the representatives Π_k $(k=0,1,2)$ are obtained form the standard base $\{\alpha_i\}_{i=1,2}$ by odd reflections. Here, we denote the standard and the other two bases by $\Pi_0=\{\alpha_i\}_{i=1,2}$, $\Pi_1=\{\beta_i\}_{i=1,2}$ and $\Pi_2=\{\gamma_i\}_{i=1,2}$. They correspond to the Dynkin diagrams:

$$\overset{\alpha_1}{\bigcirc} - \overset{\alpha_2}{\otimes}\ \underset{r_{\beta_2}}{\overset{r_{\alpha_2}}{\rightleftarrows}}\ \overset{\beta_1}{\otimes} - \overset{\beta_2}{\otimes}\ \underset{r_{\gamma_1}}{\overset{r_{\beta_1}}{\rightleftarrows}}\ \overset{\gamma_1}{\otimes} - \overset{\gamma_2}{\bigcirc} \tag{A.1}$$

and these roots satisfy $\beta_1=\alpha_1+\alpha_2$, $\beta_2=-\alpha_2$ and $\gamma_1=-\alpha_1-\alpha_2$, $\gamma_2=\alpha_1$.

Here, we use the notation in Section 2.2. We may choose the coroots $\nu^\vee := \xi^{-1}(\nu)$ for $\nu\in\Delta$. For each $x\in\{\alpha,\beta,\gamma\}$, the Cartan matrices $(\langle x_i^\vee, x_j\rangle)_{i,j=1,2}$ are given by

$$\Pi_0:\begin{pmatrix}2 & -1\\ -1 & 0\end{pmatrix},\quad \Pi_1:\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix},\quad \Pi_2:\begin{pmatrix}0 & -1\\ -1 & 2\end{pmatrix}. \tag{A.2}$$

Let $\{e_{x_i}, f_{x_i}, h \mid i\in\{1,2\}, h\in\mathfrak{h}\}$ be the Chevalley generators with commutation relations $[e_{x_i}, f_{x_j}]=\delta_{i,j}x_i^\vee$, $[h,e_{x_i}]=x_i(h)e_{x_i}$, $[h,f_{x_i}]=-x_i(h)f_{x_i}$. Further, let $\{\Lambda_{x_i}\}_{i=1,2}$ be the fundamental weights with respect to Π_k, i.e., they satisfy $\Lambda_{x_i}(x_j^\vee)=\delta_{i,j}$. These weights are explicitly given by

$$\begin{cases}\Lambda_{\alpha_1}=-\alpha_2\\ \Lambda_{\alpha_2}=-\alpha_1-2\alpha_2\end{cases},\quad \begin{cases}\Lambda_{\beta_1}=\beta_2\\ \Lambda_{\beta_2}=\beta_1,\end{cases},\quad \begin{cases}\Lambda_{\gamma_1}=-2\gamma_1-\gamma_2\\ \Lambda_{\gamma_2}=-\gamma_1\end{cases},$$

and the Weyl vectors are $\rho_{\Pi_0}=-\alpha_2$, $\rho_{\Pi_1}=0$ and $\rho_{\Pi_2}=-\gamma_1$.

We are interested in highest weight Λ such that $L_\Sigma(\Lambda)$ is an atypical integrable module. Such highest weights are given by

$$\begin{aligned}
&\Pi_0:\ \Lambda=m\Lambda_{\alpha_1}, m\Lambda_{\alpha_1}-(m+1)\Lambda_{\alpha_2}\ (m\in\mathbb{Z}_{\geq 0}),\\
&\Pi_1:\ \Lambda=m\Lambda_{\beta_1},\ m\Lambda_{\beta_2}\ (m\in\mathbb{Z}_{\geq 0}),\\
&\Pi_2:\ \Lambda=m\Lambda_{\gamma_2},\ -(m+1)\Lambda_{\gamma_1}+m\Lambda_{\gamma_2}\ (m\in\mathbb{Z}_{\geq 0}).
\end{aligned} \tag{A.3}$$

Here, we concentrate on the cases of $m\Lambda_{\alpha_1}$ and $m\Lambda_{\gamma_2}$, since

$$L_{\Pi_0}(m\Lambda_{\alpha_1} - (m+1)\Lambda_{\alpha_2}) \simeq L_{\Pi_1}((m+1)\Lambda_{\beta_2}) \simeq L_{\Pi_2}((m+1)\Lambda_{\gamma_2}),$$
$$L_{\Pi_2}(-(m+1)\Lambda_{\gamma_1} + m\Lambda_{\gamma_2}) \simeq L_{\Pi_1}((m+1)\Lambda_{\beta_1}) \simeq L_{\Pi_0}((m+1)\Lambda_{\alpha_1}).$$

Theorem A.1. *There exist the following short exact sequences of $\mathfrak{g}$-modules:*

$$\begin{aligned} 0 &\to N_{\Pi_1}(r(m\Lambda_{\alpha_1} + \rho_{\Pi_0}) - \rho_{\Pi_1}; r(\alpha_2)) \\ &\to N_{\Pi_0}(m\Lambda_{\alpha_1}; \alpha_2) \to L_{\Pi_0}(m\Lambda_{\alpha_1}) \to 0, \\ 0 &\to N_{\Pi_1}(r(m\Lambda_{\gamma_2} + \rho_{\Pi_2}) - \rho_{\Pi_1}; r(\gamma_1)) \\ &\to N_{\Pi_2}(m\Lambda_{\gamma_2}; \gamma_1) \to L_{\Pi_2}(m\Lambda_{\gamma_2}) \to 0. \end{aligned}$$

Remark A.1. Combining Theorem A.1 with Proposition 5.1, we have

$$\begin{aligned} 0 &\to N_{\Pi_2}(r(m\Lambda_{\beta_1} + \rho_{\Pi_1}) - \rho_{\Pi_2}; r(\beta_2)) \\ &\to N_{\Pi_1}(m\Lambda_{\beta_1}; \beta_2) \to L_{\Pi_1}(m\Lambda_{\beta_1}) \to 0, \\ 0 &\to N_{\Pi_0}(r_\beta(m\Lambda_{\beta_2} + \rho_{\Pi_1}) - \rho_{\Pi_0}; r_\beta(\beta_1)) \\ &\to N_{\Pi_1}(m\Lambda_{\beta_2}; \beta_1) \to L_{\Pi_1}(m\Lambda_{\beta_2}) \to 0. \end{aligned}$$

Proof of Theorem A.1. We show the theorem for $\Lambda = m\Lambda_{\alpha_1}$. Let us look at the structure of $N_{\Pi_0}(\Lambda; \alpha_2)$. For $\alpha := \alpha_1 + \alpha_2$, set $e_\alpha := [e_{\alpha_1}, e_{\alpha_2}]$, $f_\alpha := [f_{\alpha_1}, f_{\alpha_2}]$. Then, we have $[e_\alpha, f_\alpha] = \alpha^\vee$. Since $f_{\alpha_1}^{m+1}\mathbf{1}_{\Pi_0;\alpha_2}^{\Lambda}$ is a singular vector with respect to $\mathfrak{g}_{\Pi_0}^+$, there exists a homomorphism

$$\phi : M_{\Pi_0}(\Lambda - (m+1)\alpha_1) \to N_{\Pi_0}(\Lambda; \alpha_2) \quad (\mathbf{1}_{\Pi_0}^{\Lambda-(m+1)\alpha} \mapsto f_{\alpha_1}^{m+1}\mathbf{1}_{\Pi_0;\alpha_2}^{\Lambda}).$$

In order to describe the kernel of ϕ, we introduce an auxiliary module

$$\bar{M}_{\Pi_0}(\Lambda - (m+1)\alpha_1) := M_{\Pi_0}(\Lambda - (m+1)\alpha_1)/U(\mathfrak{g})f_\alpha f_{\alpha_2}\mathbf{1}_{\Pi_0}^{\Lambda-(m+1)\alpha_1}.$$

By $(f_\alpha f_{\alpha_2})f_{\alpha_1}^{m+1} = f_\alpha(f_{\alpha_1}^{m+1}f_{\alpha_2} - (m+1)f_{\alpha_1}^m f_\alpha) = f_\alpha f_{\alpha_1}^{m+1}f_{\alpha_2}$, we have $(f_\alpha f_{\alpha_2})f_{\alpha_1}^{m+1}\mathbf{1}_{\Pi_0;\alpha_2}^{\Lambda} = 0$, and thus, the map ϕ induces

$$\bar{\phi} : \bar{M}_{\Pi_0}(\Lambda - (m+1)\alpha_1) \longrightarrow N_{\Pi_0}(\Lambda; \alpha_2).$$

The odd reflection r_{α_2} gives the following isomorphism:

Lemma A.1. $\bar{M}_{\Pi_0}(\Lambda - (m+1)\alpha_1) \simeq N_{\Pi_1}(\Lambda - (m+1)\alpha_1 - \alpha_2; \alpha)$.

Proof. Since $(\Lambda-(m+1)\alpha_1,\alpha_2)\neq 0$, we have

$$M_{\Pi_1}(\Lambda-(m+1)\alpha_1-\alpha_2)\simeq M_{\Pi_0}(\Lambda-(m+1)\alpha_1) \quad (\mathbf{1}_{\Pi_1}^{\Lambda-(m+1)\alpha_1-\alpha_2}\mapsto f_{\alpha_2}\mathbf{1}_{\Pi_0}^{\Lambda-(m+1)\alpha_1}).$$

Moreover, since $(\Lambda-(m+1)\alpha_1-\alpha_2,\alpha)=0$, we obtain

$$N_{\Pi_1}(\Lambda-(m+1)\alpha_1-\alpha_2;\alpha) \simeq M_{\Pi_1}(\Lambda-(m+1)\alpha_1-\alpha_2)/U(\mathfrak{g})f_\alpha\mathbf{1}_{\Pi_1}^{\Lambda-(m+1)\alpha_1-\alpha_2}.$$

By combining these isomorphisms, the lemma is proved. □

Lemma A.2. *The map* $\bar{\phi}: N_{\Pi_1}(\Lambda-(m+1)\alpha_1-\alpha_2;\alpha)\longrightarrow N_{\Pi_0}(\Lambda;\alpha_2)$ *is injective.*

Proof. We choose $\mathbb{C}$-bases of $N_{\Pi_0}(\Lambda;\alpha_2)$ and $N_{\Pi_1}(\Lambda-(m+1)\alpha_1-\alpha_2;\alpha)$ as follows: $\{f_{\alpha_1}^k f_\alpha^l \mathbf{1}_{\Pi_0;\alpha_2}^{\Lambda}\}_{k\in\mathbb{Z}_{\geq 0},\, l\in\{0,1\}}$, $\left\{f_{\alpha_1}^k e_{\alpha_2}^l \mathbf{1}_{\Pi_1;\alpha}^{\Lambda-(m+1)\alpha_1-\alpha_2}\right\}_{k\in\mathbb{Z}_{\geq 0},\, l\in\{0,1\}}$. By $f_{\alpha_2}f_{\alpha_1}^{m+1}=f_{\alpha_1}^{m+1}f_{\alpha_2}-(m+1)f_{\alpha_1}^m f_\alpha$, we have

$$\bar{\phi}(\mathbf{1}_{\Pi_1;\alpha}^{\Lambda-(m+1)\alpha_1-\alpha_2})=f_{\alpha_2}f_{\alpha_1}^{m+1}\mathbf{1}_{\Pi_0;\alpha_2}^{\Lambda}=-(m+1)f_{\alpha_1}^m f_\alpha\mathbf{1}_{\Pi_0;\alpha_2}^{\Lambda},$$

and thus,

$$\bar{\phi}(f_{\alpha_1}^k e_{\alpha_2}^l\mathbf{1}_{\Pi_1;\alpha}^{\Lambda-(m+1)\alpha_1-\alpha_2})=\begin{cases}-(m+1)f_{\alpha_1}^{k+m}f_\alpha\mathbf{1}_{\Pi_0;\alpha_2}^{\Lambda} & (l=0)\\ (m+1)f_{\alpha_1}^{k+m+1}\mathbf{1}_{\Pi_0;\alpha_2}^{\Lambda} & (l=1)\end{cases},$$

since $[e_{\alpha_2},f_\alpha]=-f_{\alpha_1}$. The vectors in the right-hand side are linearly independent, and hence, $\bar{\phi}$ is injective. □

One can directly check that the cokernel of the map $\bar{\phi}$ is irreducible. Therefore, Theorem A.1 for $\Lambda=m\Lambda_{\alpha_1}$ holds. □

Remark A.2. Motivated by the representation theory of the $N=2$ superconformal algebra, Semikhatov and Taormina gave the BGG resolutions of atypical representations over the $\widehat{\mathfrak{sl}}(2,1)$ via certain Verma type modules called narrow Verma modules in [11]. A narrow Verma module is isomorphic to $N_\Sigma(\Lambda;\tau)$ by choosing an appropriate parabolic subalgebra $\mathfrak{p}_{\Sigma;\tau}$ of $\widehat{\mathfrak{sl}}(2,1)$ (see [6]). One of the $\mathfrak{sl}(2,1)$-counterparts of narrow Verma modules is

$$M_{\Pi_0}(\Lambda-(m+1)\alpha_1)/U(\mathfrak{g})e_{\alpha_2}f_\alpha f_{\alpha_2}\mathbf{1}_{\Pi_0}^{\Lambda-(m+1)\alpha_1}\quad(\Lambda=m\Lambda_{\alpha_1})$$

and it is isomorphic to $N_{\Pi_1}(\Lambda-(m+1)\alpha_1-\alpha_2;\alpha)$ by Lemma A.1.

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