Fixed-to-Variable Length Lossless Codes with Multiple Code Tables Considering Decoding Delay

Fixed-to-Variable Length Lossless Codes with Multiple Code Tables Considering Decoding Delay

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Abstract

Lossless source coding is a technology to represent given data in shorter bit lengths than the original representation without losing its contents. This enables efficient recording of data on storage devices and high-speed transmission of data over networks and plays an essential role in the advanced information society.

Huffman coding is a widely used lossless source coding methods in various applications such as image compression (JPEG) and video compression (H.264). The coding scheme of Huffman coding is described as the following system consisting of a *source*, an *encoder*, and a *decoder*:

- the source outputs a *source sequence*, a sequence of *source symbols* in the *source alphabet*, where each output symbol follows an independent and identical distribution;
- the encoder encodes each symbol of the source sequence to a binary *codeword* according to the *code table* obtained by Huffman's algorithm;
- then the decoder receives the *codeword sequence*, which is the concatenation of the codewords, and recovers the original source sequence from the codeword sequence.

The decoder is not given explicit information on the delimitation between the codewords in the codeword. However, the decoder can uniquely identify the delimitation by reading the codeword sequence from the beginning of it because of the following *prefix-free* property of the Huffman code table: no codeword is a prefix of any other codeword. The decoder can decode each codeword without any *decoding delay* as long as the encoder uses a prefix-free code table. For this reason, a prefix-free code is also called an *instantaneous code*. Huffman code is an instantaneous (prefix-free) code with the optimal

average codeword length, a measure of compression performance, for a given source distribution.

However, it is known that one can achieve a better compression performance than Huffman coding by using a time-variant encoder with multiple code tables and allowing some decoding delay. *AIFV (almost instantaneous fixed-to-variable length) codes* developed by Yamamoto, Tsuchihashi, and Honda can attain a smaller average codeword length than Huffman codes by using a time-variant encoder with two code tables and allowing at most 2-bit decoding delay. Moreover, AIFV codes are generalized to *AIFV-m codes*, which can achieve a smaller average codeword length than AIFV codes for $m \geq 3$, allowing m code tables and at most m-bit decoding delay.

In this thesis, we discuss a more general class of source codes with multiple code tables considering decoding delay than AIFV-m codes and show their properties. We first formalize source codes with a finite number of code tables as *code-tuples*, and then we introduce two equivalent definitions of *k-bit delay decodable code-tuples*, which allow at most k-bit decoding delay for $k \geq 0$. Then we prove three theorems related to *k-bit delay optimal code-tuples*, which are defined as code-tuples with the optimal average codeword length for a given source distribution among all the k-bit delay decodable code-tuples. These theorems describe properties of k-bit delay decodable code-tuples by the set of the possible first k bits of the codeword sequence in the case of starting from each code table.

The first theorem claims that there is no need for more than one code table such that the sets of the possible first k bits of the codeword sequence are equal. This implies that it is not the case that one can achieve an arbitrarily small average codeword length by using arbitrarily many code tables, and it is sufficient for us to consider at most finitely many code tables. In particular, this guarantees that a k-bit delay optimal code-tuple does indeed exist. Also, the first theorem gives a concrete upper bound of the required number of code tables for a k-bit delay optimal code-tuple.

The second theorem gives the following necessary condition for a k-bit delay decodable code-tuple to be optimal: if the first k bits of a given binary sequence is a prefix of some codeword sequence, then the entire given binary sequence is also a prefix of some codeword sequence. This result is a generalization of the property of Huffman codes that each internal node in the code tree has two child nodes.

The third theorem shows that it is sufficient to consider only the code-tuples such that both 0 and 1 are possible as the first bit of codeword no matter which code table we start the encoding process from.

These three theorems enable us to limit the scope of codes to be considered when discussing k-bit delay optimal codes in theoretical analysis and practical code construction.

As applications of the three theorems, for $k = 1, 2$, we give a class of k-bit delay decodable code-tuples which include a k-bit delay optimal code-tuple for a given source distribution. More specifically, we first prove that the Huffman code achieves the optimal average codeword length in the class of 1-bit delay decodable code-tuples. Namely, the class of instantaneous codes with a single code table can achieve the optimal average codeword length in the class of 1-bit delay decodable code-tuples. Then we also prove that the class of AIFV codes can achieve the optimal average codeword length in the class of 2-bit delay decodable code-tuples. In particular, this result implies that it is sufficient to consider at most two code tables to find a 2-bit delay optimal code-tuple.

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Chapter 1

Introduction

1.1 Background

1.1.1 Data Compression

Data compression is a technology to represent given data in shorter bit lengths than the original representation without compromising its utility. This enables efficient recording of data on storage devices and high-speed transmission of data over networks. Therefore, data compression technology plays an essential role in the advanced information society, where vast amounts of data are processed on computers and transmitted over networks.

Data compression is studied as *source coding* in *information theory*, a mathematical field which deals with information mathematically, established by Claude E. Shannon [1]. Source coding is divided into *lossy source coding*, where some of the original information is lost by the compression process, and *lossless source coding*, where the original information is fully preserved. Huffman coding, stated later in Subsection 1.1.4, is an example of the widely used lossless source coding methods in various applications such as image compression (JPEG) and video compression (H.264). This thesis focuses on lossless source coding.

1.1.2 Source Coding

Lossless (binary) source coding is modeled as a process of encoding a given *source sequence* of *source symbols* of the *source alphabet* $\mathcal{S}$ to a *codeword sequence* over the binary *coding alphabet* $\mathcal{C} \coloneqq \{0, 1\}$ and then recovering the original source sequence from the codeword sequence; that is, encoding an element $\boldsymbol{x}$ of $\mathcal{S}^*$ to an element $\boldsymbol{b}$ of $\mathcal{C}^*$ temporally and then recovering $\boldsymbol{x}$ from $\boldsymbol{b}$ later, where $\mathcal{S}^*$ and $\mathcal{C}^*$ denote the set of all sequences of finite length over $\mathcal{S}$ and $\mathcal{C}$, respectively. We now describe one of the simplest models of source coding, which consists of a *source*, an *encoder*, and a *decoder* as follows.

- Source: outputs a source sequence $\boldsymbol{x} = x_1 x_2 \ldots x_n \in \mathcal{S}^*$, where each output x_i independently follows a fixed probability distribution $\mu : \mathcal{S} \to (0, 1]$, that is, $\mu(s)$ is the probability of occurrence of $s \in \mathcal{S}$.

- Encoder: reads the source sequence $\boldsymbol{x} = x_1 x_2 \ldots x_n \in \mathcal{S}^*$ symbol by symbol from the beginning and encodes each symbol x_i to a *codeword* $f(x_i) \in \mathcal{C}^*$ according to a fixed code table $f : \mathcal{S} \to \mathcal{C}^*$. Then the encoder outputs the concatenation of each codeword as a codeword sequence, that is, the encoder outputs $f^*(x_1 x_2 \ldots x_n) \coloneqq f(x_1) f(x_2) \ldots f(x_n)$ for the given source sequence $\boldsymbol{x} = x_1 x_2 \ldots x_n \in \mathcal{S}^*$.

- Decoder: receives the codeword sequence $f^*(\boldsymbol{x})$ and recovers the original source sequence $\boldsymbol{x} \in \mathcal{S}^*$ from $f^*(\boldsymbol{x})$. Note that for the decoder to be able to correctly recover $\boldsymbol{x}$, the code table f must be chosen so that the mapping $f^* : \mathcal{S}^* \to \mathcal{C}^*$ is injective.

The behavior of the encoder and decoder is determined by the code table f. Accordingly, we refer to a mapping $f : \mathcal{S} \to \mathcal{C}^*$ as a *source code* or a *code* and identify the encoder and decoder with f.

The compression performance of a code f is evaluated by the *average codeword length* $L^\mu(f)$ defined as the expected value of the length of the codeword.

Definition 1.1.1. *The* average codeword length $L^\mu(f)$ *of a code* f *(with respect to* μ*) is defined as*

$$L^\mu(f) = \sum_{s \in \mathcal{S}} \mu(s) |f(s)|, \tag{1.1}$$

where $|f(s)|$ *denotes the length of* $f(s)$*.*

Then the following problem naturally arises: how good (small) average codeword length can be achieved under the constraint that the decoder can correctly recover the original source sequence?

1.1.3 Uniquely Decodable Codes

A *uniquely decodable* code is a source code f such that the decoder can correctly recover the original source sequence.

Definition 1.1.2. *A code f is said to be* uniquely decodable *if the mapping f^* is injective.*

It is known that a lower bound of achievable average codeword length of uniquely decodable codes is given by the *entropy* of μ.

Theorem 1.1.1 ([1]). *For any uniquely decodable code f, we have*

$$L^\mu(f) \geq H(\mu), \tag{1.2}$$

where $H(\mu)$ is the entropy *of μ defined as*

$$H(\mu) = -\sum_{s\in\mathcal{S}} \mu(s) \log_2 \mu(s). \tag{1.3}$$

Namely, uniquely decodable codes cannot achieve an average code length smaller than the entropy. Therefore, the next focus is how close average codeword length to the entropy can be achieved, that is, how small *redundancy* η, defined as follows, can be achieved:

$$\eta := L^\mu(f) - H(\mu). \tag{1.4}$$

By the following McMillan's theorem, we can always achieve a redundancy smaller than or equal to 1.

Theorem 1.1.2 (McMillan's inequality [2]). *Let $\mathcal{S} = \{s_1, s_2, \ldots, s_\sigma\}$. Let $(l_1, l_2, \ldots, l_\sigma)$ be a sequence of non-negative integers. Then the following two conditions (a) and (b) are equivalent.*

(a) There exists a uniquely decodable code f such that $|f(s_i)| = l_i$ for any $i = 1, 2, \ldots, \sigma$.

(b) $\sum_{i=1}^{\sigma} 2^{-l_i} \leq 1$.

Indeed, if we choose $(l_1, l_2, \ldots, l_\sigma)$ as $l_i = \lceil -\log_2 \mu(s_i) \rceil$ for $i = 1, 2, \ldots, \sigma$, then

$$\sum_{i=1}^{\sigma} 2^{-l_i} = \sum_{i=1}^{\sigma} 2^{-\lceil -\log_2 \mu(s_i)\rceil} \leq \sum_{i=1}^{\sigma} 2^{\log_2 \mu(s_i)} = \sum_{i=1}^{\sigma} \mu(s_i) = 1 \tag{1.5}$$

and thus by Theorem 1.1.2, there exists a uniquely decodable code f such that

$$L^{\mu}(f) = \sum_{i=1}^{\sigma} \mu(s_i) \lceil -\log_2 \mu(s_i) \rceil \le \sum_{i=1}^{\sigma} \mu(s_i)(-\log_2 \mu(s_i) + 1) = H(\mu) + 1. \tag{1.6}$$

Namely, for any μ, there exists a uniquely decodable code f with a redundancy $L^{\mu}(f) - H(\mu) \le 1$. In other words, the *worst-case redundancy* of the class of uniquely decodable codes, defined as follows, is less than or equal to 1.

Definition 1.1.3. *Let $\mathcal{C}$ be a set of codes. Then the* worst-case redundancy *of $\mathcal{C}$ is*

$$\sup_{\mu \in \mathscr{M}} \left(\inf_{f \in \mathcal{C}} \left(L^{\mu}(f) - H(\mu) \right) \right), \tag{1.7}$$

where $\mathscr{M}$ denotes the set of all probability distributions.

Conversely, for a source alphabet $\mathcal{S} = \{s_1, s_2, \ldots, s_\sigma\}$ with $\sigma \ge 2$, the worst-case redundancy of the class of uniquely decodable codes is not less than 1 for the following reason. For any uniquely decodable code f and probability distribution μ, we have $L^{\mu}(f) \ge 1$ since each codeword must be non-empty. On the other hand, for any $\epsilon > 0$, the probability distribution μ defined as

$$(\mu(s_1), \mu(s_2), \ldots, \mu(s_\sigma)) = \left(\delta, \frac{1-\delta}{\sigma-1}, \frac{1-\delta}{\sigma-1}, \ldots, \frac{1-\delta}{\sigma-1} \right) \tag{1.8}$$

satisfies $H(\mu) < \epsilon$ for a sufficiently large $0 < \delta < 1$. Therefore, for any $\epsilon > 0$, there exists a probability distribution μ such that

$$L^{\mu}(f) - H(\mu) > 1 - \epsilon \tag{1.9}$$

holds for any uniquely decodable code f.

1.1.4 Prefix-free Codes

The decoder receives a codeword sequence $f^*(x_1 x_2 \ldots x_n)$, which is the concatenation of individual codewords $f(x_1), f(x_2), \ldots, f(x_n)$, without explicit information on the delimitation between the codewords. Therefore, even if f

is uniquely decodable, the original source sequence $\boldsymbol{x}$ can be recovered only after the decoder has read the entire codeword sequence in the worst case. However, if f is a *prefix-free* code defined as below, then when the decoder reads the codeword sequence from the beginning, it can instantly identify the delimitation of each codeword at the moment the decoder reaches the end of each codeword.

Definition 1.1.4. *A code f is said to be* prefix-free *if for any $s, s' \in \mathcal{S}$, if $f(s) \preceq f(s')$, then $s = s'$, where $\boldsymbol{x} \preceq \boldsymbol{y}$ denotes that $\boldsymbol{x}$ is a prefix of $\boldsymbol{y}$.*

In other words, a code is prefix-free if and only if no codeword is a prefix of any other codeword. A prefix-free code is also called an *instantaneous code* because the decoder can identify the delimitation instantly as mentioned above.

Regarding prefix-codes, the following Kraft's Theorem holds.

Theorem 1.1.3 (Kraft's inequality [3])**.** *Let $\mathcal{S} = \{s_1, s_2, \ldots, s_\sigma\}$. Let $(l_1, l_2, \ldots, l_\sigma)$ be a sequence of non-negative integers. Then the following two conditions (a) and (b) are equivalent.*

(a) There exists a prefix-free code f such that $|f(s_i)| = l_i$ for any $i = 1, 2, \ldots, \sigma$.

(b) $\sum_{i=1}^{\sigma} 2^{-l_i} \leq 1$.

Comparing Theorems 1.1.2 and 1.1.3, we see that for any sequence $(l_1, l_2, \ldots, l_\sigma)$, the following equivalent relation holds: there exists a uniquely decodable code f such that $|f(s_i)| = l_i$ if and only if there exists a prefix-free code f' such that $|f'(s_i)| = l_i$. Moreover, the average codeword length of a code f is determined only by the multiset of codeword lengths $|f(s_1)|, |f(s_2)|, \ldots, |f(s_\sigma)|$. This yields the following result.

Theorem 1.1.4. *For any uniquely decodable code f, there exists a prefix-free code f' such that $L^\mu(f') = L^\mu(f)$.*

In this sense, it is sufficient to consider only the class of prefix-free codes instead of the whole class of uniquely decodable codes. Then the next question is how to give a prefix-free code with the minimum average codeword length among all prefix-free codes. Huffman [4] gave an algorithm to construct a prefix-free code with the optimal average codeword length for a given source distribution μ. The source code obtained by Huffman's algorithm is called *Huffman code*. Huffman codes achieve the optimal average codeword length in the class of prefix-free codes. By Theorem 1.1.4, Huffman codes are also optimal in the class of uniquely decodable codes.

However, in the discussion so far, we assumed that a single code table is used for coding. It is known that one can achieve a smaller average codeword length than Huffman codes by using a time-variant encoder with multiple code tables as mentinoned in the next subsection.

1.1.5 AIFV-m Codes

AIFV (almost instantaneous fixed-to-variable length) codes [5] developed by Yamamoto, Tsuchihashi, and Honda can attain a smaller average codeword length than Huffman codes by using a time-variant encoder with a pair (f_0, f_1) of two code tables and allowing at most 2-bit decoding delay. Further, AIFV-m codes [6], which is a generalization of AIFV codes, can achieve a smaller average codeword length than AIFV codes for $m \geq 3$ by allowing a tuple $(f_0, f_1, \ldots, f_{m-1})$ of m code tables and at most m-bit decoding delay (the original AIFV codes [5] are particular cases of AIFV-m codes for $m = 2$). The worst-case redundancy of AIFV-m codes is given as $1/m$ for $m \leq 5$ as shown in [6, 7].

The literature [8–16] gives the construction methods of the optimal AIFV codes and AIFV-m codes for a given source distribution. The coding methods of code tables $(f_0, f_1, \ldots, f_{m-1})$ of AIFV and AIFV-m codes are studied in [17, 18]. Extensions of AIFV-m codes are proposed in [19, 20]. Other relevant studies include [21–23].

1.2 Contribution

The proposal for AIFV-m codes motivates us to study source codes with multiple code tables considering some decoding delay. This thesis discusses a more general class of such source codes than AIFV-m codes: we introduce a notion of *code-tuple*, which is a source code with a finite number of code tables, and we investigate the properties of the class of *k-bit delay decodable code-tuples*, which are code-tuples allowing at most k-bit decoding delay for $k \geq 0$.

- We prove three theorems related to *k-bit delay optimal code-tuples*, which are code-tuples achieving the optimal average codeword length for a given source distribution among all the k-bit delay decodable code-tuples.

  - The first theorem gives an upper bound of the required number of code tables for a k-bit delay optimal code-tuples. This shows that it is not the case that one can achieve an arbitrarily small average codeword length by using arbitrarily many code tables, in particular, the existence of a k-bit delay optimal code-tuple.
  - The second theorem gives the following necessary condition for a k-bit delay decodable code-tuple to be optimal: if the first k bits of a given binary sequence is a prefix of some codeword sequence, then the entire given binary sequence is also a prefix of some codeword sequence. This is a generalization of the property of Huffman codes that each internal node in the code tree has two child nodes.
  - The third theorem shows the existence of an optimal k-bit delay decodable code-tuple F such that both $0, 1 \in \mathcal{C}$ are possible as the first bit of codeword no matter which code table of F we start the encoding process from.

  These theorems enable us to limit the scope of codes to be considered when discussing k-bit delay optimal code-tuples.

- As applications of the three theorems, we give a class of code-tuples which can achieve the optimal average codeword length in the class of k-bit delay decodable code-tuples for $k = 1, 2$.
  - We prove that the Huffman code achieves the optimal average codeword length in the class of 1-bit delay decodable code-tuples for a given source distribution μ.
  - We also prove that the optimal AIFV codes can achieve the optimal average codeword length in the class of 2-bit delay decodable code-tuples for a given source distribution μ.

1.3 Organization

This thesis is organized as follows.

- In Chapter 2, we prepare some notations, describe our data compression scheme, and basic definitions, and show their properties.
  - In Section 2.1, we formalize source codes with a finite number of code tables as *code-tuples*.
  - In Section 2.2, we state two equivalent definitions of the class $\mathscr{F}_{k\text{-dec}}$ of k-bit delay decodable code-tuples in Section 2.2.
  - In Section 2.3, we introduce a class $\mathscr{F}_{\mathrm{ext}}$ of *extendable* code-tuples, to exclude some abnormal code-tuples from consideration.
  - In Section 2.4, we define the average codeword length of a code-tuple and introduce a class F_{reg} of *regular* code-tuples.
  - In Section 2.5, we define a class $\mathscr{F}_{\mathrm{irr}}$ of *irreducible* code-tuples and introduce *irreducible parts* of a code-tuple, which are obtained by removing the non-essential code tables from a code-tuple.
- In Chapter 3, we introduce a class $\mathscr{F}_{k\text{-opt}}$ of *k-bit delay optimal* code-tuples and prove three theorems on the general properties of k-bit delay optimal code-tuples as parts of the main results.
  - In Section 3.1, we first explain the statements of the three theorems.
  - In Section 3.2–3.4, we give the proofs of the three theorems, respectively.
- In Chapter 4, we present a class of code-tuples which can achieve the optimal average codeword length in the class of k-bit delay decodable code-tuples for $k = 1, 2$.
  - In Section 4.1, we prove that the class of Huffman codes achieves the optimal average codeword length in the class of 1-bit delay decodable code-tuples.
  - In Section 4.2, we prove that the class of AIFV codes achieves the optimal average codeword length in the class of 2-bit delay decodable code-tuples.
- In Chapter 5, we summarize our results and conclude with future works.

The main notations are listed in Appendix A. To clarify the flow of discussion, we relegate long proofs to the end of each chapter.

Chapter 2

Preliminaries

We define some notations as follows. Let $\mathbb{R}$ denote the set of all real numbers, and let $\mathbb{R}^m$ denote the set of all m-dimensional real row vectors for an integer $m \geq 1$. Let $|\mathcal{A}|$ denote the cardinality of a finite set $\mathcal{A}$. Let $\mathcal{A} \times \mathcal{B}$ denote the Cartesian product of $\mathcal{A}$ and $\mathcal{B}$, that is, $\mathcal{A} \times \mathcal{B} \coloneqq \{(a, b) : a \in \mathcal{A}, b \in \mathcal{B}\}$. Let $\mathcal{A}^k$ (resp. $\mathcal{A}^{\leq k}$, $\mathcal{A}^{\geq k}$, $\mathcal{A}^*$, $\mathcal{A}^+$) denote the set of all sequences of length k (resp. of length less than or equal to k, of length greater than or equal to k, of finite length, of finite positive length) over a set $\mathcal{A}$. Thus, $\mathcal{A}^+ = \mathcal{A}^* \setminus \{\lambda\}$, where λ denotes the empty sequence. The length of a sequence $\boldsymbol{x}$ is denoted by $|\boldsymbol{x}|$, in particular, $|\lambda| = 0$. The i-th letter of a sequence $\boldsymbol{x}$ is denoted by x_i. For a non-empty sequence $\boldsymbol{x} = x_1 x_2 \ldots x_n$, we define $\mathrm{pref}(\boldsymbol{x}) \coloneqq x_1 x_2 \ldots x_{n-1}$ and $\mathrm{suff}(\boldsymbol{x}) \coloneqq x_2 \ldots x_{n-1} x_n$. Namely, $\mathrm{pref}(\boldsymbol{x})$ (resp. $\mathrm{suff}(\boldsymbol{x})$) is the sequence obtained by deleting the last (resp. first) letter from $\boldsymbol{x}$. We say $\boldsymbol{x} \preceq \boldsymbol{y}$ if $\boldsymbol{x}$ is a prefix of $\boldsymbol{y}$, that is, there exists a sequence $\boldsymbol{z}$, possibly $\boldsymbol{z} = \lambda$, such that $\boldsymbol{y} = \boldsymbol{x}\boldsymbol{z}$. Also, we say $\boldsymbol{x} \prec \boldsymbol{y}$ if $\boldsymbol{x} \preceq \boldsymbol{y}$ and $\boldsymbol{x} \neq \boldsymbol{y}$. Moreover, we say $\boldsymbol{x} \not\precsim \boldsymbol{y}$ if $\boldsymbol{x} \not\preceq \boldsymbol{y}$ and $\boldsymbol{x} \not\succeq \boldsymbol{y}$. A notation $\boldsymbol{x} \wedge \boldsymbol{y}$ denotes the longest common prefix of two sequences $\boldsymbol{x}$ and $\boldsymbol{y}$, that is, the longest sequence $\boldsymbol{z}$ such that $\boldsymbol{z} \preceq \boldsymbol{x}$ and $\boldsymbol{z} \preceq \boldsymbol{y}$. If $\boldsymbol{x} \preceq \boldsymbol{y}$, then $\boldsymbol{x}^{-1}\boldsymbol{y}$ denotes the unique sequence $\boldsymbol{z}$ such that $\boldsymbol{x}\boldsymbol{z} = \boldsymbol{y}$. Note that a notation $\boldsymbol{x}^{-1}$ behaves like the "inverse element" of $\boldsymbol{x}$ as stated in the following statements (i)–(iii).

(i) For any $\boldsymbol{x}$, we have $\boldsymbol{x}^{-1}\boldsymbol{x} = \lambda$.

(ii) For any $\boldsymbol{x}$ and $\boldsymbol{y}$ such that $\boldsymbol{x} \preceq \boldsymbol{y}$, we have $\boldsymbol{x}\boldsymbol{x}^{-1}\boldsymbol{y} = \boldsymbol{y}$.

(iii) For any $\boldsymbol{x}, \boldsymbol{y}$, and $\boldsymbol{z}$ such that $\boldsymbol{x}\boldsymbol{y} \preceq \boldsymbol{z}$, we have $(\boldsymbol{x}\boldsymbol{y})^{-1}\boldsymbol{z} = \boldsymbol{y}^{-1}\boldsymbol{x}^{-1}\boldsymbol{z}$.

For $c \in \mathcal{C}$, the negation of c is denoted by $\bar{c}$, that is, $\bar{0} := 1, \bar{1} := 0$. Also, for $\boldsymbol{c} \in \mathcal{C}^{\geq k}$, let $[\boldsymbol{c}]_k$ denote the prefix of length k of $\boldsymbol{c}$. Moreover, for $\boldsymbol{c} \in \mathcal{C}^*$ and $\mathcal{A} \subseteq \mathcal{C}^*$, we define $\boldsymbol{c}\mathcal{A} := \{\boldsymbol{cb} : \boldsymbol{b} \in \mathcal{A}\}$. The main notations used in this thesis are listed in Appendix A.

We now describe our data compression system. In this thesis, we consider a data compression system consisting of a source, an encoder, and a decoder.

- Source: We consider an i.i.d.(independent and identical distribution) source, which outputs a sequence $\boldsymbol{x} = x_1 x_2 \ldots x_n$ of symbols of the source alphabet $\mathcal{S} = \{s_1, s_2, \ldots, s_\sigma\}$, where n and σ denote the length of $\boldsymbol{x}$ and the alphabet size, respectively. Each source output follows a fixed probability distribution $(\mu(s_1), \mu(s_2), \ldots, \mu(s_\sigma))$, where $\mu(s_i)$ is the probability of occurrence of s_i for $i = 1, 2, \ldots, \sigma$. In this thesis we assume that the alphabet size $\sigma = |\mathcal{S}|$ is greater than or equal to 2.

- Encoder: The encoder has m fixed code tables $f_0, f_1, \ldots, f_{m-1} : \mathcal{S} \to \mathcal{C}^*$, where $\mathcal{C} = \{0, 1\}$ is the coding alphabet. The encoder reads the source sequence $\boldsymbol{x} \in \mathcal{S}^*$ symbol by symbol from the beginning of $\boldsymbol{x}$ and encodes them according to the code tables. For the first symbol x_1, we use an arbitrarily chosen code table from $f_0, f_1, \ldots, f_{m-1}$. For $x_2, x_3, \ldots, x_n$, we determine which code table to use to encode according to m fixed mappings $\tau_0, \tau_1, \ldots, \tau_{m-1} : \mathcal{S} \to [m] := \{0, 1, 2, \ldots, m-1\}$. More specifically, if the previous symbol x_{i-1} is encoded by the code table f_j, then the current symbol x_i is encoded by the code table $f_{\tau_j(x_{i-1})}$. Hence, if we use the code table f_i to encode x_1, then a source sequence $\boldsymbol{x} = x_1 x_2 \ldots x_n$ is encoded to a codeword sequence $f(\boldsymbol{x}) := f_{i_1}(x_1) f_{i_2}(x_2) \ldots f_{i_n}(x_n)$, where

$$i_j := \begin{cases} i & \text{if } j = 1, \\ \tau_{i_{j-1}}(x_{j-1}) & \text{if } j \geq 2 \end{cases} \tag{2.1}$$

  for $j = 1, 2, \ldots, n$.

- Decoder: The decoder reads the codeword sequence $f(\boldsymbol{x})$ bit by bit from the beginning of $f(\boldsymbol{x})$. Each time the decoder reads a bit, the decoder recovers as long prefix of $\boldsymbol{x}$ as the decoder can uniquely identify from the prefix of $f(\boldsymbol{x})$ already read. We assume that the encoder and decoder share the index of the code table used to encode x_1 in advance.

2.1 Code-tuples

In our data compression system described above, the behavior of the encoder and decoder for a given source sequence is completely determined by m code tables $f_0, f_1, \ldots, f_{m-1}$ and m mappings $\tau_0, \tau_1, \ldots, \tau_{m-1}$ if we fix the index of code table used to encode x_1. Accordingly, we name a tuple $F(f_0, f_1, \ldots, f_{m-1}, \tau_0, \tau_1, \ldots, \tau_{m-1})$ as a *code-tuple* F in the following Definition 2.1.1, and we identify a source code with a code-tuple F.

Definition 2.1.1. *Let m be a positive integer. An m-*code-tuple *$F(f_0, f_1, \ldots, f_{m-1}, \tau_0, \tau_1, \ldots, \tau_{m-1})$ is a tuple of m mappings $f_0, f_1, \ldots, f_{m-1} : \mathcal{S} \to \mathcal{C}^*$ and m mappings $\tau_0, \tau_1, \ldots, \tau_{m-1} : \mathcal{S} \to [m]$.*

We define $\mathscr{F}^{(m)}$ as the set of all m-code-tuples. Also, we define $\mathscr{F} \coloneqq \mathscr{F}^{(1)} \cup \mathscr{F}^{(2)} \cup \mathscr{F}^{(3)} \cup \cdots$. An element of $\mathscr{F}$ is called a code-tuple.

We write $F(f_0, f_1, \ldots, f_{m-1}, \tau_0, \tau_1, \ldots, \tau_{m-1})$ also as $F(f, \tau)$ or F for simplicity. For $F \in \mathscr{F}^{(m)}$, let $|F|$ denote the number of code tables of F, that is, $|F| \coloneqq m$. We write $[|F|] = \{0, 1, 2, \ldots, |F| - 1\}$ as $[F]$ for simplicity.

Definition 2.1.2. *For $F(f, \tau) \in \mathscr{F}, i \in [F]$, and $\boldsymbol{b} \in \mathcal{C}^*$, we define*

$$\mathcal{S}_{F,i}(\boldsymbol{b}) \coloneqq \{s \in \mathcal{S} : f_i(s) = \boldsymbol{b}\}. \tag{2.2}$$

Note that f_i is injective if and only if $|\mathcal{S}_{F,i}(\boldsymbol{b})| \leq 1$ holds for any $\boldsymbol{b} \in \mathcal{C}^*$.

Example 2.1.1. *Table 2.1 shows examples of a 3-code-tuple $F^{(\alpha)}, F^{(\beta)}, F^{(\gamma)}$ for $\mathcal{S} = \{\mathrm{a}, \mathrm{b}, \mathrm{c}, \mathrm{d}\}$. We have*

$$\mathcal{S}_{F^{(\alpha)},0}(110) = \{\mathrm{a}, \mathrm{c}\}, \quad \mathcal{S}_{F^{(\beta)},1}(00000000) = \emptyset, \quad \mathcal{S}_{F^{(\alpha)},2}(\lambda) = \{\mathrm{a}, \mathrm{b}, \mathrm{c}, \mathrm{d}\}. \tag{2.3}$$

Example 2.1.2. *We consider encoding of a source sequence $\boldsymbol{x} = x_1x_2x_3x_4 \coloneqq \mathrm{badb}$ with the code-tuple $F(f, \tau) \coloneqq F^{(\gamma)}$ in Table 2.1. If $x_1 = \mathrm{b}$ is encoded with the code table f_0, then the encoding process is as follows.*

- *$x_1 = \mathrm{b}$ is encoded to $f_0(\mathrm{b}) = 10$. The index of the next code table is $\tau_0(\mathrm{b}) = 1$.*
- *$x_2 = \mathrm{a}$ is encoded to $f_1(\mathrm{a}) = 00$. The index of the next code table is $\tau_1(\mathrm{a}) = 1$.*

Table 2.1: Examples of a code-tuple $F^{(\alpha)}, F^{(\beta)}, F^{(\gamma)}$

$s \in \mathcal{S}$	$f_0^{(\alpha)}$	$\tau_0^{(\alpha)}$	$f_1^{(\alpha)}$	$\tau_1^{(\alpha)}$	$f_2^{(\alpha)}$	$\tau_2^{(\alpha)}$
a	110	0	010	0	λ	2
b	λ	1	011	2	λ	2
c	110	2	1	2	λ	2
d	111	0	10	1	λ	2

$s \in \mathcal{S}$	$f_0^{(\beta)}$	$\tau_0^{(\beta)}$	$f_1^{(\beta)}$	$\tau_1^{(\beta)}$	$f_2^{(\beta)}$	$\tau_2^{(\beta)}$
a	11	1	0110	1	10	2
b	λ	1	0110	1	11	2
c	101	2	01	1	1000	2
d	1011	1	0111	1	1001	2

$s \in \mathcal{S}$	$f_0^{(\gamma)}$	$\tau_0^{(\gamma)}$	$f_1^{(\gamma)}$	$\tau_1^{(\gamma)}$	$f_2^{(\gamma)}$	$\tau_2^{(\gamma)}$
a	01	0	00	1	1100	1
b	10	1	λ	0	1110	0
c	0100	0	00111	1	111000	2
d	01	2	00111	2	110	2

- $x_3 = \mathrm{d}$ *is encoded to* $f_1(\mathrm{d}) = 00111$. *The index of the next code table is* $\tau_1(\mathrm{d}) = 2$.

- $x_4 = \mathrm{b}$ *is encoded to* $f_2(\mathrm{b}) = 1110$. *The index of the next code table is* $\tau_2(\mathrm{b}) = 0$.

As the result, we obtain a codeword sequence $\boldsymbol{c} := f_0(\mathrm{b})f_1(\mathrm{a})f_1(\mathrm{d})f_2(\mathrm{b}) = 1000001111110$.

The decoding process of $\boldsymbol{c} = 1000001111110$ *is as follows.*

- *After reading the prefix* 10 *of* $\boldsymbol{c}$, *the decoder can uniquely identify* $x_1 = \mathrm{b}$ *and* $10 = f_0(\mathrm{b})$. *The decoder can also know that* x_2 *is decoded with* $f_{\tau_0(\mathrm{b})} = f_1$.

- *After reading the prefix* $1000 = f_0(\mathrm{c})f_1(\mathrm{a})$ *of* $\boldsymbol{c}$, *the decoder still cannot uniquely identify* $x_2 = \mathrm{a}$ *because there remain three possible cases: the case* $x_2 = \mathrm{a}$, *the case* $x_2 = \mathrm{c}$, *and the case* $x_2 = \mathrm{d}$.

- *After reading the prefix* 10000 *of* $\boldsymbol{c}$, *the decoder can uniquely identify* $x_2 = \mathrm{a}$ *and* $10000 = f_0(\mathrm{b})f_1(\mathrm{a})0$. *The decoder can also know that* x_3 *is* *decoded with* $f_{\tau_1(\mathrm{a})} = f_1$.

- *After reading the prefix* $100000111 = f_0(\mathrm{b})f_1(\mathrm{a})f_1(\mathrm{d})$ *of* $\boldsymbol{c}$*, the decoder still cannot uniquely identify* $x_3 = \mathrm{d}$ *because there remain two possible cases: the case* $x_3 = \mathrm{c}$ *and the case* $x_3 = \mathrm{d}$.

- *After reading the prefix* 10000011111 *of* $\boldsymbol{c}$*, the decoder can uniquely identify* $x_3 = \mathrm{d}$ *and* $10000011111 = f_0(\mathrm{b})f_1(\mathrm{a})f_1(\mathrm{d})11$*. The decoder can also know that* x_4 *is decoded with* $f_{\tau_1(\mathrm{d})} = f_2$.

- *After reading the prefix* $\boldsymbol{c} = 1000001111110$*, the decoder can uniquely identify* $x_4 = \mathrm{b}$ *and* $1000001111110 = f_0(\mathrm{b})f_1(\mathrm{a})f_1(\mathrm{d})f_2(\mathrm{b})$.

As the result, the decoder recovers the original sequence $\boldsymbol{x} = \mathrm{badb}$.

In encoding $\boldsymbol{x} = x_1x_2\ldots x_n \in \mathcal{S}^*$ with a code-tuple $F(f,\tau)$, the m mappings $\tau_0, \tau_1, \ldots, \tau_{m-1}$ determine which code table to use to encode $x_2, x_3, \ldots,$ x_n. However, there are choices of which code table to use for the first symbol x_1. For $i \in [F]$ and $\boldsymbol{x} \in \mathcal{S}^*$, we define $f_i^*(\boldsymbol{x}) \in \mathcal{C}^*$ as the codeword sequence in the case where x_1 is encoded with f_i. Also, we define $\tau_i^*(\boldsymbol{x}) \in [F]$ as the index of the code table used next after encoding $\boldsymbol{x}$ in the case where x_1 is encoded with f_i. We give formal definitions of f_i^* and τ_i^* in the following Definition 2.1.3 as recursive formulas.

Definition 2.1.3. *For* $F(f,\tau) \in \mathscr{F}$ *and* $i \in [F]$*, we define a mapping* $f_i^* : \mathcal{S}^* \to \mathcal{C}^*$ *and a mapping* $\tau_i^* : \mathcal{S}^* \to [F]$ *as*

$$f_i^*(\boldsymbol{x}) = \begin{cases} \lambda & \text{if } \boldsymbol{x} = \lambda, \\ f_i(x_1)f^*_{\tau_i(x_1)}(\mathrm{suff}(\boldsymbol{x})) & \text{if } \boldsymbol{x} \neq \lambda, \end{cases} \tag{2.4}$$

$$\tau_i^*(\boldsymbol{x}) = \begin{cases} i & \text{if } \boldsymbol{x} = \lambda, \\ \tau^*_{\tau_i(x_1)}(\mathrm{suff}(\boldsymbol{x})) & \text{if } \boldsymbol{x} \neq \lambda \end{cases} \tag{2.5}$$

for $\boldsymbol{x} = x_1x_2\ldots x_n \in \mathcal{S}^*$.

Example 2.1.3. *We consider* $F(f,\tau) \coloneqq F^{(\gamma)}(f^{(\gamma)}, \tau^{(\gamma)})$ *of Table 2.1. Then*

$f_0^*(\text{badb})$ *and* $\tau_0^*(\text{badb})$ *is given as follows (cf. Example 2.1.2):*

$$\begin{aligned} f_0^*(\text{badb}) &= f_0(\text{b})f_1^*(\text{adb}) && (2.6)\\ &= f_0(\text{b})f_1(\text{a})f_1^*(\text{db}) && (2.7)\\ &= f_0(\text{b})f_1(\text{a})f_1(\text{d})f_2^*(\text{b}) && (2.8)\\ &= f_0(\text{b})f_1(\text{a})f_1(\text{d})f_2(\text{b})f_0^*(\lambda) && (2.9)\\ &= 1000001111110, && (2.10)\end{aligned}$$

$$\tau_0^*(\text{badb}) = \tau_1^*(\text{adb}) = \tau_1^*(\text{db}) = \tau_2^*(\text{b}) = \tau_0^*(\lambda) = 0. \qquad (2.11)$$

The following Lemma 2.1.1 follows from Definition 2.1.3.

Lemma 2.1.1. *For any* $F(f,\tau) \in \mathscr{F}$*,* $i \in [F]$*, and* $\boldsymbol{x}, \boldsymbol{y} \in \mathcal{S}^*$*, the following statements (i)–(iii) hold.*

(i) $f_i^*(\boldsymbol{xy}) = f_i^*(\boldsymbol{x})f^*_{\tau_i^*(\boldsymbol{x})}(\boldsymbol{y})$.

(ii) $\tau_i^*(\boldsymbol{xy}) = \tau^*_{\tau_i^*(\boldsymbol{x})}(\boldsymbol{y})$.

(iii) If $\boldsymbol{x} \preceq \boldsymbol{y}$*, then* $f_i^*(\boldsymbol{x}) \preceq f_i^*(\boldsymbol{y})$.

2.2 k-bit Delay Decodable Code-tuples

In Example 2.1.2, despite $f_0(\text{b})f_1(\text{a}) = 1000$, to uniquely identify $x_1x_2 = \text{ba}$, it is required to read 10000 including the additional 1 bit. Namely, a decoding delay of 1 bit occurs to decode $x_2 = \text{a}$. Similarly, despite $f_0(\text{b})f_0(\text{a})f_1(\text{d}) = 100000111$, to uniquely identify $x_1x_2x_3 = \text{bad}$, it is required to read 10000011111 including the additional 2 bits. Namely, a decoding delay of 2 bits occurs to decode $x_3 = \text{d}$. In general, in the decoding process with $F^{(\gamma)}$ in Table 2.1, it is required to read the additional at most 2 bits for the decoder to uniquely identify each symbol of a given source sequence. We say a code-tuple is *k-bit delay decodable* if the decoder can always uniquely identify each source symbol by reading the additional k bits of the codeword sequence. The code-tuple $F^{(\gamma)}$ is an example of a 2-bit delay decodable code-tuple. In this section, we state two equivalent formal definitions of a k-bit delay decodable code-tuple.

2.2.1 The First Definition

To state the first definition of a k-bit delay decodable code-tuple, we first introduce the following Definitions 2.2.1 and 2.2.2.

Definition 2.2.1. *For an integer $k \geq 0$, $F(f,\tau) \in \mathscr{F}, i \in [F]$, and $\boldsymbol{b} \in \mathcal{C}^*$, we define*

$$\mathcal{P}^k_{F,i}(\boldsymbol{b}) \coloneqq \{\boldsymbol{c} \in \mathcal{C}^k : \boldsymbol{x} \in \mathcal{S}^+, f^*_i(\boldsymbol{x}) \succeq \boldsymbol{bc}, f_i(x_1) \succeq \boldsymbol{b}\}, \tag{2.12}$$

$$\bar{\mathcal{P}}^k_{F,i}(\boldsymbol{b}) \coloneqq \{\boldsymbol{c} \in \mathcal{C}^k : \boldsymbol{x} \in \mathcal{S}^+, f^*_i(\boldsymbol{x}) \succeq \boldsymbol{bc}, f_i(x_1) \succ \boldsymbol{b}\}. \tag{2.13}$$

Namely, $\mathcal{P}^k_{F,i}(\boldsymbol{b})$ (resp. $\bar{\mathcal{P}}^k_{F,i}(\boldsymbol{b})$) is the set of all $\boldsymbol{c} \in \mathcal{C}^k$ such that there exists $\boldsymbol{x} = x_1x_2\ldots x_n \in \mathcal{S}^+$ satisfying $f^*_i(\boldsymbol{x}) \succeq \boldsymbol{bc}$ and $f_i(x_1) \succeq \boldsymbol{b}$ (resp. $f_i(x_1) \succ \boldsymbol{b}$).

Definition 2.2.2. *For $F(f,\tau) \in \mathscr{F}, i \in [F]$, and $\boldsymbol{b} \in \mathcal{C}^*$, we define*

$$\mathcal{P}^*_{F,i}(\boldsymbol{b}) \coloneqq \mathcal{P}^0_{F,i}(\boldsymbol{b}) \cup \mathcal{P}^1_{F,i}(\boldsymbol{b}) \cup \mathcal{P}^2_{F,i}(\boldsymbol{b}) \cup \cdots, \tag{2.14}$$

$$\bar{\mathcal{P}}^*_{F,i}(\boldsymbol{b}) \coloneqq \bar{\mathcal{P}}^0_{F,i}(\boldsymbol{b}) \cup \bar{\mathcal{P}}^1_{F,i}(\boldsymbol{b}) \cup \bar{\mathcal{P}}^2_{F,i}(\boldsymbol{b}) \cup \cdots. \tag{2.15}$$

We write $\mathcal{P}^k_{F,i}(\lambda)$ (resp. $\bar{\mathcal{P}}^k_{F,i}(\lambda)$) as $\mathcal{P}^k_{F,i}$ (resp. $\bar{\mathcal{P}}^k_{F,i}$) for simplicity. Also, we write $\mathcal{P}^*_{F,i}(\lambda)$ (resp. $\bar{\mathcal{P}}^*_{F,i}(\lambda)$) as $\mathcal{P}^*_{F,i}$ (resp. $\bar{\mathcal{P}}^*_{F,i}$). We have

$$\mathcal{P}^k_{F,i} \overset{\text{(A)}}{=} \{\boldsymbol{c} \in \mathcal{C}^k : \boldsymbol{x} \in \mathcal{S}^+, f^*_i(\boldsymbol{x}) \succeq \boldsymbol{c}\} \overset{\text{(B)}}{=} \{\boldsymbol{c} \in \mathcal{C}^k : \boldsymbol{x} \in \mathcal{S}^*, f^*_i(\boldsymbol{x}) \succeq \boldsymbol{c}\}, \tag{2.16}$$

where (A) follows from (2.12), and (B) is justified as follows. The relation "$\subseteq$" holds by $\mathcal{S}^+ \subseteq \mathcal{S}^*$. We show the relation "$\supseteq$". We choose $\boldsymbol{c} \in \mathcal{C}^k$ such that $f^*_i(\boldsymbol{x}) \succeq \boldsymbol{c}$ for some $\boldsymbol{x} \in \mathcal{S}^*$ arbitrarily and show that $f^*_i(\boldsymbol{x}') \succeq \boldsymbol{c}$ for some $\boldsymbol{x}' \in \mathcal{S}^+$. The case $\boldsymbol{x} \in \mathcal{S}^+$ is trivial. In the case $\boldsymbol{x} \in \{\lambda\} = \mathcal{S}^* \setminus \mathcal{S}^+$, then since $\boldsymbol{c} \preceq f^*_i(\boldsymbol{x}) = f^*_i(\lambda) = \lambda$ by (2.4), we have $\boldsymbol{c} = \lambda$, which leads to that any $\boldsymbol{x}' \in \mathcal{S}^+$ satisfies $f^*_i(\boldsymbol{x}') \succeq \lambda = \boldsymbol{c}$. Hence, the relation "$\supseteq$" holds.

Example 2.2.1. *We consider $F(f,\tau) \coloneqq F^{(\beta)}$ in Table 2.1. First, we confirm $\mathcal{P}^3_{F,0}(\boldsymbol{b}) = \{100, 101, 111\}$ for $\boldsymbol{b} = 101$ as follows.*

- $100 \in \mathcal{P}^3_{F,0}(\boldsymbol{b})$ *holds because* $\boldsymbol{x} = \mathrm{cc}$ *satisfies* $f^*_0(\boldsymbol{x}) = 1011000 \succeq \boldsymbol{b}100$ *and* $f_0(x_1) = 101 \succeq \boldsymbol{b}$.
- $101 \in \mathcal{P}^3_{F,0}(\boldsymbol{b})$ *holds because* $\boldsymbol{x} = \mathrm{da}$ *satisfies* $f^*_0(\boldsymbol{x}) = 10110110 \succeq \boldsymbol{b}101$ *and* $f_0(x_1) = 1011 \succeq \boldsymbol{b}$.

Table 2.2: The set $\mathcal{P}^1_{F,i}$ and $\mathcal{P}^2_{F,i}$ for the code-tuples F in Table 2.1

$F \in \mathscr{F}$	$\mathcal{P}^1_{F,0}$	$\mathcal{P}^1_{F,1}$	$\mathcal{P}^1_{F,2}$	$\mathcal{P}^2_{F,0}$	$\mathcal{P}^2_{F,1}$	$\mathcal{P}^2_{F,2}$
$F^{(\alpha)}$	$\{0,1\}$	$\{0,1\}$	$\emptyset$	$\{01,10,11\}$	$\{01,10\}$	$\emptyset$
$F^{(\beta)}$	$\{0,1\}$	$\{0\}$	$\{1\}$	$\{01,10,11\}$	$\{01\}$	$\{10, 11\}$
$F^{(\gamma)}$	$\{0,1\}$	$\{0,1\}$	$\{1\}$	$\{01,10\}$	$\{00,01,10\}$	$\{11\}$

Table 2.3: The set $\bar{\mathcal{P}}^2_{F,i}(f_i(s))$ for $F := F^{(\gamma)}$

$s \in \mathcal{S}$	$\bar{\mathcal{P}}^2_{F,0}(f_0(s))$	$\bar{\mathcal{P}}^2_{F,1}(f_1(s))$	$\bar{\mathcal{P}}^2_{F,2}(f_2(s))$
a	$\{00\}$	$\{11\}$	$\emptyset$
b	$\emptyset$	$\{00\}$	$\{00\}$
c	$\emptyset$	$\emptyset$	$\emptyset$
d	$\{00\}$	$\emptyset$	$\{00,01\}$

- $111 \in \mathcal{P}^3_{F,0}(\boldsymbol{b})$ *holds because* $\boldsymbol{x} = \mathrm{cbb}$ *satisfies* $f^*_0(\boldsymbol{x}) = 1011111 \succeq \boldsymbol{b}111$ *and* $f_0(x_1) = 101 \succeq \boldsymbol{b}$.

Next, we confirm $\bar{\mathcal{P}}^3_{F,0}(\boldsymbol{b}) = \{101\}$ *for* $\boldsymbol{b} = 101$ *as follows.*

- $101 \in \bar{\mathcal{P}}^3_{F,0}(\boldsymbol{b})$ *holds because* $\boldsymbol{x} = \mathrm{da}$ *satisfies* $f^*_0(\boldsymbol{x}) = 10110110 \succeq \boldsymbol{b}101$ *and* $f_0(x_1) = 1011 \succ \boldsymbol{b}$.

Also, we confirm $\bar{\mathcal{P}}^0_{F,1}(\boldsymbol{b}) = \{\lambda\}$ *for* $\boldsymbol{b} = 011$ *as follows.*

- $\lambda \in \bar{\mathcal{P}}^0_{F,1}(\boldsymbol{b})$ *holds because* $\boldsymbol{x} = \mathrm{a}$ *satisfies* $f^*_1(\boldsymbol{x}) = 0110 \succeq \boldsymbol{b} = \boldsymbol{b}\lambda$ *and* $f_1(x_1) = 0110 \succ \boldsymbol{b}$.

Example 2.2.2. *Table 2.2 shows* $\mathcal{P}^1_{F,i}$ *and* $\mathcal{P}^2_{F,i}$ *for the code-tuples* F *in Table 2.1. Also, Table 2.3 shows* $\bar{\mathcal{P}}^2_{F,i}(f_i(s))$ *for* $F(f,\tau) := F^{(\gamma)}$ *in Table 2.1.*

We show the basic properties of the sets $\mathcal{P}^k_{F,i}(\boldsymbol{b})$ and $\bar{\mathcal{P}}^k_{F,i}(\boldsymbol{b})$ as the following Lemmas 2.2.1 and 2.2.2.

Lemma 2.2.1. *For any integer* $k \geq 0$, $F(f,\tau) \in \mathscr{F}$, $i \in [F]$, *and* $\boldsymbol{b} \in \mathcal{C}^*$, *the following statements (i) and (ii) hold.*

(i)

$$\mathcal{P}^k_{F,i}(\boldsymbol{b}) = \bar{\mathcal{P}}^k_{F,i}(\boldsymbol{b}) \cup \Big(\bigcup_{s \in \mathcal{S}_{F,i}(\boldsymbol{b})} \mathcal{P}^k_{F,\tau_i(s)} \Big). \tag{2.17}$$

(ii) If $k \geq 1$, then

$$\bar{\mathcal{P}}^k_{F,i}(\boldsymbol{b}) = 0\mathcal{P}^{k-1}_{F,i}(\boldsymbol{b}0) \cup 1\mathcal{P}^{k-1}_{F,i}(\boldsymbol{b}1). \tag{2.18}$$

Proof of Lemma 2.2.1. (Proof of (i)): For any $\boldsymbol{c} \in \mathcal{C}^k$, we have

$$\begin{aligned}
&\boldsymbol{c} \in \mathcal{P}^k_{F,i}(\boldsymbol{b}) \\
&\overset{\text{(A)}}{\Longleftrightarrow} \quad {}^\exists \boldsymbol{x} \in \mathcal{S}^+; (f^*_i(\boldsymbol{x}) \succeq \boldsymbol{bc}, f_i(x_1) \succeq \boldsymbol{b}) && (2.19)\\
&\Longleftrightarrow \quad ({}^\exists \boldsymbol{x} \in \mathcal{S}^+; (f^*_i(\boldsymbol{x}) \succeq \boldsymbol{bc}, f_i(x_1) \succ \boldsymbol{b})) \\
&\qquad\quad \text{or } ({}^\exists \boldsymbol{x} \in \mathcal{S}^+; (f^*_i(\boldsymbol{x}) \succeq \boldsymbol{bc}, f_i(x_1) = \boldsymbol{b})) && (2.20)\\
&\overset{\text{(B)}}{\Longleftrightarrow} \quad \boldsymbol{c} \in \bar{\mathcal{P}}^k_{F,i}(\boldsymbol{b}) \text{ or } {}^\exists \boldsymbol{x} \in \mathcal{S}^+; (f^*_i(\boldsymbol{x}) \succeq \boldsymbol{bc}, f_i(x_1) = \boldsymbol{b}) && (2.21)\\
&\overset{\text{(C)}}{\Longleftrightarrow} \quad \boldsymbol{c} \in \bar{\mathcal{P}}^k_{F,i}(\boldsymbol{b}) \text{ or } {}^\exists \boldsymbol{x} \in \mathcal{S}^+; (f^*_{\tau_i(x_1)}(\mathrm{suff}(\boldsymbol{x})) \succeq \boldsymbol{c}, f_i(x_1) = \boldsymbol{b}) && (2.22)\\
&\Longleftrightarrow \quad \boldsymbol{c} \in \bar{\mathcal{P}}^k_{F,i}(\boldsymbol{b}) \text{ or } {}^\exists s \in \mathcal{S}; {}^\exists \boldsymbol{x} \in \mathcal{S}^*; (f^*_{\tau_i(s)}(\boldsymbol{x}) \succeq \boldsymbol{c}, f_i(s) = \boldsymbol{b}) && (2.23)\\
&\Longleftrightarrow \quad \boldsymbol{c} \in \bar{\mathcal{P}}^k_{F,i}(\boldsymbol{b}) \text{ or } {}^\exists s \in \mathcal{S}_{F,i}(\boldsymbol{b}); {}^\exists \boldsymbol{x} \in \mathcal{S}^*; f^*_{\tau_i(s)}(\boldsymbol{x}) \succeq \boldsymbol{c} && (2.24)\\
&\overset{\text{(D)}}{\Longleftrightarrow} \quad \boldsymbol{c} \in \bar{\mathcal{P}}^k_{F,i}(\boldsymbol{b}) \text{ or } {}^\exists s \in \mathcal{S}_{F,i}(\boldsymbol{b}); \boldsymbol{c} \in \mathcal{P}^k_{F,\tau_i(s)} && (2.25)\\
&\Longleftrightarrow \quad \boldsymbol{c} \in \bar{\mathcal{P}}^k_{F,i}(\boldsymbol{b}) \text{ or } \boldsymbol{c} \in \bigcup_{s \in \mathcal{S}_{F,i}(\boldsymbol{b})} \mathcal{P}^k_{F,\tau_i(s)} && (2.26)\\
&\Longleftrightarrow \quad \boldsymbol{c} \in \bar{\mathcal{P}}^k_{F,i}(\boldsymbol{b}) \cup \Big(\bigcup_{s \in \mathcal{S}_{F,i}(\boldsymbol{b})} \mathcal{P}^k_{F,\tau_i(s)} \Big) && (2.27)
\end{aligned}$$

as desired, where (A) follows from (2.12), (B) follows from (2.13), (C) follows from (2.4), and (D) follows from (2.16).

(Proof of (ii)): For any $\boldsymbol{c} \in \mathcal{C}^k$, we have

$$\begin{aligned}
&\boldsymbol{c} \in \bar{\mathcal{P}}^k_{F,i}(\boldsymbol{b}) \\
&\overset{\text{(A)}}{\Longleftrightarrow} {}^\exists \boldsymbol{x} \in \mathcal{S}^+; (f^*_i(\boldsymbol{x}) \succeq \boldsymbol{bc}, f_i(x_1) \succ \boldsymbol{b}) && (2.28)\\
&\Longleftrightarrow {}^\exists \boldsymbol{x} \in \mathcal{S}^+; (f^*_i(\boldsymbol{x}) \succeq \boldsymbol{b}c_1\mathrm{suff}(\boldsymbol{c}), f_i(x_1) \succeq \boldsymbol{b}c_1) && (2.29)\\
&\Longleftrightarrow (c_1 = 0, {}^\exists \boldsymbol{x} \in \mathcal{S}^+; (f^*_i(\boldsymbol{x}) \succeq \boldsymbol{b}0\mathrm{suff}(\boldsymbol{c}), f_i(x_1) \succeq \boldsymbol{b}0)) \\
&\qquad \text{or } (c_1 = 1, {}^\exists \boldsymbol{x} \in \mathcal{S}^+; (f^*_i(\boldsymbol{x}) \succeq \boldsymbol{b}1\mathrm{suff}(\boldsymbol{c}), f_i(x_1) \succeq \boldsymbol{b}1)) && (2.30)\\
&\overset{\text{(B)}}{\Longleftrightarrow} (c_1 = 0, \mathrm{suff}(\boldsymbol{c}) \in \mathcal{P}^{k-1}_{F,i}(\boldsymbol{b}0)) \text{ or } (c_1 = 1, \mathrm{suff}(\boldsymbol{c}) \in \mathcal{P}^{k-1}_{F,i}(\boldsymbol{b}1)) && (2.31)\\
&\Longleftrightarrow \boldsymbol{c} \in 0\mathcal{P}^{k-1}_{F,i}(\boldsymbol{b}0) \text{ or } \boldsymbol{c} \in 1\mathcal{P}^{k-1}_{F,i}(\boldsymbol{b}1) && (2.32)\\
&\Longleftrightarrow \boldsymbol{c} \in 0\mathcal{P}^{k-1}_{F,i}(\boldsymbol{b}0) \cup 1\mathcal{P}^{k-1}_{F,i}(\boldsymbol{b}1) && (2.33)
\end{aligned}$$

as desired, where (A) follows from (2.13), and (B) follows from (2.12). □

Lemma 2.2.2. *For any $F(f,\tau) \in \mathscr{F}$ and $i \in [F]$, the following statements (i)–(iii) hold.*

(i) For any $\boldsymbol{b} \in \mathcal{C}^*$, the following equivalence relation holds: $\bar{\mathcal{P}}^0_{F,i}(\boldsymbol{b}) \neq \emptyset \iff {}^\exists s \in \mathcal{S}; f_i(s) \succ \boldsymbol{b}$.

(ii) There exists $s \in \mathcal{S}$ such that $\bar{\mathcal{P}}^0_{F,i}(f_i(s)) = \emptyset$.

(iii) If $|\mathcal{S}_{F,i}(\lambda)| \leq 1$, in particular f_i is injective, then $\bar{\mathcal{P}}^0_{F,i} \neq \emptyset$.

Proof of Lemma 2.2.2. (Proof of (i)): We have

$$\lambda \in \bar{\mathcal{P}}^0_{F,i}(\boldsymbol{b}) \overset{\text{(A)}}{\iff} {}^\exists \boldsymbol{x} \in \mathcal{S}^+; (f_i^*(\boldsymbol{x}) \succeq \boldsymbol{b}, f_i(x_1) \succ \boldsymbol{b}) \iff {}^\exists s \in \mathcal{S}; f_i(s) \succ \boldsymbol{b} \tag{2.34}$$

as desired, where (A) follows from (2.13).

(Proof of (ii)): Let $s \in \arg\max\{|f_i(s')| : s' \in \mathcal{S}\}$. Then there is no $s' \in \mathcal{S}$ such that $f_i(s) \prec f_i(s')$. Hence, by (i) of this lemma, we obtain $\bar{\mathcal{P}}^0_{F,i}(f_i(s)) = \emptyset$.

(Proof of (iii)): By $|\mathcal{S}_{F,i}(\lambda)| \leq 1$ and the assumption that $|\mathcal{S}| \geq 2$, there exists $s \in \mathcal{S}$ such that $f_i(s) \neq \lambda$. This is equivalent to $\bar{\mathcal{P}}^0_{F,i} \neq \emptyset$ by (i) of this lemma. □

We consider the situation where the decoder has already read the prefix $\boldsymbol{b}'$ of a given codeword sequence and identified a prefix $x_1x_2\ldots x_l$ of the original sequence $\boldsymbol{x}$. Then we have $\boldsymbol{b}' = f_{i_1}(x_1)f_{i_2}(x_2)\ldots f_{i_l}(x_l)\boldsymbol{b}$ for some $\boldsymbol{b} \in \mathcal{C}^*$. Put $i := i_{l+1}$ and let $\{s_1, s_2, \ldots, s_r\}$ be the set of all symbols $s \in \mathcal{S}$ such that $f_i(s) = \boldsymbol{b}$. Then there are the following $r+1$ possible cases for the next symbol x_{l+1}: the case $x_{l+1} = s_1$, the case $x_{l+1} = s_2$, $\ldots$, the case $x_{l+1} = s_r$, and the case $f_i(x_{l+1}) \succ \boldsymbol{b}$. For a code-tuple F to be k-bit delay decodable, the decoder must always be able to distinguish these $r+1$ cases by reading the following k bits of the codeword sequence. Namely, it is required that the $r+1$ sets listed below are disjoint:

- $\mathcal{P}^k_{F,\tau_i(s_1)}$, the set of all possible following k bits in the case $x_{l+1} = s_1$,
- $\mathcal{P}^k_{F,\tau_i(s_2)}$, the set of all possible following k bits in the case $x_{l+1} = s_2$,
- $\cdots$,
- $\mathcal{P}^k_{F,\tau_i(s_r)}$, the set of all possible following k bits in the case $x_{l+1} = s_r$,

- $\bar{\mathcal{P}}^k_{F,i}(\boldsymbol{b})$, the set of all possible following k bits in the case $f_i(x_{l+1}) \succ \boldsymbol{b}$.

Example 2.2.3. *We obtain* $f_0^*(\boldsymbol{x}) = 1000001111110$ *by encoding* $\boldsymbol{x} := \mathrm{badb}$ *with* $F(f,\tau) := F^{(\gamma)}$ *in Table 2.1 (cf. Example 2.1.2). We consider the decoding process of* $f_0^*(\boldsymbol{x})$*.*

- *First, we suppose that the decoder already read the prefix* $\boldsymbol{b}' = 1000$ *of* $f_0^*(\boldsymbol{x})$ *and identified* $x_1 = \mathrm{b}$*. Then we have* $\boldsymbol{b}' = f_0(x_1)00$ *and* $\mathcal{S}_{F,1}(00) = \{\mathrm{a}\}$*, and the next symbol* x_2 *is decoded with* $f_{\tau_0(\mathrm{b})} = f_1$*. Now, there are two possible cases for* x_2*: the case* $x_2 = \mathrm{a}$ *and the case* $f_1(x_2) \succ 00$ *(i.e.,* $x_2 = \mathrm{c}$ *or* $x_2 = \mathrm{d}$*). The decoder can distinguish these two cases by reading the following* 2 *bits because*

  - $\mathcal{P}^2_{F,\tau_1(\mathrm{a})}$*, the set of all possible following* 2 *bits in the case* $x_2 = \mathrm{a}$*, and*
  - $\bar{\mathcal{P}}^2_{F,1}(00)$*, the set of all possible following* 2 *bits in the case* $f_1(x_2) \succ \boldsymbol{b}$*,*

  are disjoint: $\mathcal{P}^2_{F,\tau_1(\mathrm{a})} \cap \bar{\mathcal{P}}^2_{F,1}(f_1(\mathrm{a})) = \{00, 01, 10\} \cap \{11\} = \emptyset$*. Since the following* 2 *bits are* $00 \in \mathcal{P}^2_{F,\tau_1(\mathrm{a})}$*, the decoder can identify* $x_2 = \mathrm{a}$ *indeed.*

- *Next, we suppose that the decoder already read the prefix* $\boldsymbol{b}' = 100000$ *of* $f_0^*(\boldsymbol{x})$ *and identified* $x_1x_2 = \mathrm{ba}$*. Then we have* $\boldsymbol{b}' = f_0^*(x_1x_2)00$ *and* $\mathcal{S}_{F,1}(00) = \{\mathrm{a}\}$*, and the next symbol* x_3 *is decoded with* $f_{\tau_1(\mathrm{a})} = f_1$*. Now, there are two possible cases for* x_3*: the case* $x_3 = \mathrm{a}$ *and the case* $f_1(x_3) \succ 00$ *(i.e.,* $x_3 = \mathrm{c}$ *or* $x_3 = \mathrm{d}$*). The decoder can distinguish these two cases by reading the following* 2 *bits because*

  - $\mathcal{P}^2_{F,\tau_1(\mathrm{a})}$*, the set of all possible following* 2 *bits in the case* $x_3 = \mathrm{a}$*, and*
  - $\bar{\mathcal{P}}^2_{F,1}(00)$*, the set of all possible following* 2 *bits in the case* $f_1(x_3) \succ \boldsymbol{b}$*,*

  are disjoint: $\mathcal{P}^2_{F,\tau_1(\mathrm{a})} \cap \bar{\mathcal{P}}^2_{F,1}(f_1(\mathrm{a})) = \{00, 01, 10\} \cap \{11\} = \emptyset$*. Since the following* 2 *bits are* $11 \in \bar{\mathcal{P}}^2_{F,1}(00)$*, the decoder can identify* $f_1(x_3) \succ 00$*, in particular,* $x_3 \neq \mathrm{a}$ *indeed.*

- *Lastly, we suppose that the decoder already read the prefix* $\boldsymbol{b}' = 100000111$ *of* $f_0^*(\boldsymbol{x})$ *and identified* $x_1x_2 = \mathrm{ba}$. *Then we have* $\boldsymbol{b}' = f_0^*(\mathrm{ba})00111$ *and* $\mathcal{S}_{F,1}(00111) = \{\mathrm{c}, \mathrm{d}\}$. *Now, there are two possible cases for* x_3*: the case* $x_3 = \mathrm{c}$ *and the case* $x_3 = \mathrm{d}$. *The decoder can distinguish these two cases by reading the following* 2 *bits because*

  - $\mathcal{P}^2_{F,\tau_1(\mathrm{c})}$*, the set of all possible following* 2 *bits in the case* $x_2 = \mathrm{c}$*, and*
  - $\mathcal{P}^2_{F,\tau_1(\mathrm{d})}$*, the set of all possible following* 2 *bits in the case* $x_2 = \mathrm{d}$*,*

  are disjoint: $\mathcal{P}^2_{F,\tau_1(\mathrm{c})} \cap \mathcal{P}^2_{F,\tau_1(\mathrm{d})} = \{00, 01, 10\} \cap \{11\} = \emptyset$. *Since the following* 2 *bits are* $11 \in \mathcal{P}^2_{F,\tau_1(\mathrm{d})}$*, the decoder can identify* $x_3 = \mathrm{d}$ *indeed.*

Based on the discussion above, the first definition of a k-bit delay decodable code is given as the following Definition 2.2.3.

Definition 2.2.3. *Let* $k \geq 0$ *be an integer. A code-tuple* $F(f, \tau)$ *is said to be* k-bit delay decodable *if the following conditions (a) and (b) hold.*

(a) For any $i \in [F]$ *and* $s \in \mathcal{S}$*, it holds that* $\mathcal{P}^k_{F,\tau_i(s)} \cap \bar{\mathcal{P}}^k_{F,i}(f_i(s)) = \emptyset$.

(b) For any $i \in [F]$ *and* $s, s' \in \mathcal{S}$*, if* $s \neq s'$ *and* $f_i(s) = f_i(s')$*, then* $\mathcal{P}^k_{F,\tau_i(s)} \cap \mathcal{P}^k_{F,\tau_i(s')} = \emptyset$.

For an integer $k \geq 0$*, we define* $\mathscr{F}_{k\text{-dec}}$ *as the set of all* k*-bit delay decodable code-tuples, that is,*

$$\mathscr{F}_{k\text{-dec}} := \{F \in \mathscr{F} : F \text{ is } k\text{-bit delay decodable}\}. \tag{2.35}$$

Example 2.2.4. *We confirm* $F(f, \tau) := F^{(\gamma)}$ *in Table 2.1 is* 2*-bit delay decodable as follows.*

First, we see that F *satisfies Definition 2.2.3 (a) as follows (cf. Tables 2.2 and 2.3).*

- $\mathcal{P}^2_{F,\tau_0(\mathrm{a})} \cap \bar{\mathcal{P}}^2_{F,0}(f_0(\mathrm{a})) = \mathcal{P}^2_{F,0} \cap \bar{\mathcal{P}}^2_{F,0}(f_0(\mathrm{a})) = \{01, 10\} \cap \{00\} = \emptyset$.
- $\mathcal{P}^2_{F,\tau_0(\mathrm{b})} \cap \bar{\mathcal{P}}^2_{F,0}(f_0(\mathrm{b})) = \mathcal{P}^2_{F,1} \cap \bar{\mathcal{P}}^2_{F,0}(f_0(\mathrm{b})) = \{00, 01, 10\} \cap \emptyset = \emptyset$.
- $\mathcal{P}^2_{F,\tau_0(\mathrm{c})} \cap \bar{\mathcal{P}}^2_{F,0}(f_0(\mathrm{c})) = \mathcal{P}^2_{F,0} \cap \bar{\mathcal{P}}^2_{F,0}(f_0(\mathrm{c})) = \{01, 10\} \cap \emptyset = \emptyset$.

- $\mathcal{P}^2_{F,\tau_0(\mathrm{d})} \cap \bar{\mathcal{P}}^2_{F,0}(f_0(\mathrm{d})) = \mathcal{P}^2_{F,2} \cap \bar{\mathcal{P}}^2_{F,0}(f_0(\mathrm{d})) = \{11\} \cap \{00\} = \emptyset.$
- $\mathcal{P}^2_{F,\tau_1(\mathrm{a})} \cap \bar{\mathcal{P}}^2_{F,1}(f_1(\mathrm{a})) = \mathcal{P}^2_{F,1} \cap \bar{\mathcal{P}}^2_{F,1}(f_1(\mathrm{a})) = \{00, 01, 10\} \cap \{11\} = \emptyset.$
- $\mathcal{P}^2_{F,\tau_1(\mathrm{b})} \cap \bar{\mathcal{P}}^2_{F,1}(f_1(\mathrm{b})) = \mathcal{P}^2_{F,0} \cap \bar{\mathcal{P}}^2_{F,1}(f_1(\mathrm{b})) = \{01, 10\} \cap \{00\} = \emptyset.$
- $\mathcal{P}^2_{F,\tau_1(\mathrm{c})} \cap \bar{\mathcal{P}}^2_{F,1}(f_1(\mathrm{c})) = \mathcal{P}^2_{F,1} \cap \bar{\mathcal{P}}^2_{F,1}(f_1(\mathrm{c})) = \{00, 01, 10\} \cap \emptyset = \emptyset.$
- $\mathcal{P}^2_{F,\tau_1(\mathrm{d})} \cap \bar{\mathcal{P}}^2_{F,1}(f_1(\mathrm{d})) = \mathcal{P}^2_{F,2} \cap \bar{\mathcal{P}}^2_{F,1}(f_1(\mathrm{d})) = \{11\} \cap \emptyset = \emptyset.$
- $\mathcal{P}^2_{F,\tau_2(\mathrm{a})} \cap \bar{\mathcal{P}}^2_{F,2}(f_2(\mathrm{a})) = \mathcal{P}^2_{F,1} \cap \bar{\mathcal{P}}^2_{F,2}(f_2(\mathrm{a})) = \{00, 01, 10\} \cap \emptyset = \emptyset.$
- $\mathcal{P}^2_{F,\tau_2(\mathrm{b})} \cap \bar{\mathcal{P}}^2_{F,2}(f_2(\mathrm{b})) = \mathcal{P}^2_{F,0} \cap \bar{\mathcal{P}}^2_{F,2}(f_2(\mathrm{b})) = \{01, 10\} \cap \{00\} = \emptyset.$
- $\mathcal{P}^2_{F,\tau_2(\mathrm{c})} \cap \bar{\mathcal{P}}^2_{F,2}(f_2(\mathrm{c})) = \mathcal{P}^2_{F,2} \cap \bar{\mathcal{P}}^2_{F,2}(f_2(\mathrm{c})) = \{11\} \cap \emptyset = \emptyset.$
- $\mathcal{P}^2_{F,\tau_2(\mathrm{d})} \cap \bar{\mathcal{P}}^2_{F,2}(f_2(\mathrm{d})) = \mathcal{P}^2_{F,2} \cap \bar{\mathcal{P}}^2_{F,2}(f_2(\mathrm{d})) = \{11\} \cap \{00, 01\} = \emptyset.$

Next, we see that F satisfies Definition 2.2.3 (b) as follows (cf. Table 2.2).

- $\mathcal{P}^2_{F,\tau_0(\mathrm{a})} \cap \mathcal{P}^2_{F,\tau_0(\mathrm{d})} = \mathcal{P}^2_{F,0} \cap \mathcal{P}^2_{F,2} = \{01, 10\} \cap \{11\} = \emptyset.$
- $\mathcal{P}^2_{F,\tau_1(\mathrm{c})} \cap \mathcal{P}^2_{F,\tau_1(\mathrm{d})} = \mathcal{P}^2_{F,1} \cap \mathcal{P}^2_{F,2} = \{00, 01, 10\} \cap \{11\} = \emptyset.$

Consequently, we have $F \in \mathscr{F}_{2\text{-dec}}$.

Example 2.2.5. *In a similar way to Example 2.2.4, we can see that the code-tuple $F^{(\alpha)}$ in Table 2.1 is 2-bit delay decodable. We give some more examples as follows.*

- *For $F(f, \tau) \coloneqq F^{(\alpha)}$, we have $F \notin \mathscr{F}_{1\text{-dec}}$ because $\mathcal{P}^1_{F,\tau_0(\mathrm{b})} \cap \bar{\mathcal{P}}^1_{F,0}(f_0(\mathrm{b})) = \{0, 1\} \cap \{1\} = \{1\} \neq \emptyset$.*
- *For $F(f, \tau) \coloneqq F^{(\beta)}$, for any integer $k \geq 0$, we have $F \notin \mathscr{F}_{k\text{-dec}}$ because $\mathcal{P}^k_{F,\tau_1(\mathrm{a})} \cap \mathcal{P}^k_{F,\tau_1(\mathrm{b})} = \mathcal{P}^k_{F,1} \cap \mathcal{P}^k_{F,1} = \mathcal{P}^k_{F,1} \neq \emptyset$.*
- *For $F(f, \tau) \coloneqq F^{(\gamma)}$, we have $F \notin \mathscr{F}_{1\text{-dec}}$ because $\mathcal{P}^1_{F,\tau_1(\mathrm{c})} \cap \mathcal{P}^1_{F,\tau_1(\mathrm{d})} = \{0, 1\} \cap \{1\} = \{1\} \neq \emptyset$.*

Remark 2.2.1. *If all the code tables $f_0, f_1, \ldots, f_{|F|-1}$ are injective, then Definition 2.2.3 (b) holds since there are no $i \in [F]$ and $s, s' \in \mathcal{S}$ such that $s \neq s$ and $f_i(s) \neq f_i(s')$.*

If $k = 0$, then the converse also holds as seen below. We consider Definition 2.2.3 (b) for the case $k = 0$. Then by (2.16), we have $\mathcal{P}^k_{F,\tau_i(s)} \cap \mathcal{P}^k_{F,\tau_i(s')} = \{\lambda\} \cap \{\lambda\} = \{\lambda\} \neq \emptyset$ for any $i \in [F]$ and $s, s' \in \mathcal{S}$. Hence, for F to satisfy Definition 2.2.3 (b), it is required that for any $i \in [F]$ and $s, s' \in \mathcal{S}$, if $s \neq s'$, then $f_i(s) \neq f_i(s')$, that is, $f_0, f_1, \ldots, f_{|F|-1}$ are injective.

Remark 2.2.2. *A k-bit delay decodable code-tuple F is not necessarily uniquely decodable, that is, the mappings $f^*_0, f^*_1, \ldots, f^*_{|F|-1}$ are not necessarily injective. For example, for $F^{(\gamma)} \in \mathscr{F}_{2\text{-dec}}$ in Table 2.1, we have ${f^{(\gamma)}_0}^*(\mathrm{bc}) = 1000111 = {f^{(\gamma)}_0}^*(\mathrm{bd})$. In general, it is possible that the decoder cannot uniquely recover the last few symbols of the original source sequence in the case where the rest of the codeword sequence is less than k bits. In such a case, we should append additional information for practical use.*

For k-bit delay decodable code-tuples, the following Lemma 2.2.3 holds.

Lemma 2.2.3. *For any integer $k \geq 0$, $F(f, \tau) \in \mathscr{F}_{k\text{-dec}}$, $i \in [F]$, and $\boldsymbol{b} \in \mathcal{C}^*$, we have*

$$|\mathcal{P}^k_{F,i}(\boldsymbol{b})| = |\bar{\mathcal{P}}^k_{F,i}(\boldsymbol{b})| + \sum_{s \in \mathcal{S}_{F,i}(\boldsymbol{b})} |\mathcal{P}^k_{F,\tau_i(s)}|. \tag{2.36}$$

Proof of Lemma 2.2.3. We have

$$|\mathcal{P}^k_{F,i}(\boldsymbol{b})| \overset{\text{(A)}}{=} |\bar{\mathcal{P}}^k_{F,i}(\boldsymbol{b}) \cup \Big(\bigcup_{s \in \mathcal{S}_{F,i}(\boldsymbol{b})} \mathcal{P}^k_{F,\tau_i(s)} \Big)| \tag{2.37}$$

$$\overset{\text{(B)}}{=} |\bar{\mathcal{P}}^k_{F,i}(\boldsymbol{b})| + |\bigcup_{s \in \mathcal{S}_{F,i}(\boldsymbol{b})} \mathcal{P}^k_{F,\tau_i(s)}| \tag{2.38}$$

$$\overset{\text{(C)}}{=} |\bar{\mathcal{P}}^k_{F,i}(\boldsymbol{b})| + \sum_{s \in \mathcal{S}_{F,i}(\boldsymbol{b})} |\mathcal{P}^k_{F,\tau_i(s)}| \tag{2.39}$$

as desired, where (A) follows from Lemma 2.2.1 (i), (B) follows from $F \in \mathscr{F}_{k\text{-dec}}$ and Definition 2.2.3 (a), and (C) follows from $F \in \mathscr{F}_{k\text{-dec}}$ and Definition 2.2.3 (b). □

2.2.2 The Second Definition

We give the second definition of a k-bit delay decodable codes. We first fix $F(f,\tau) \in \mathscr{F}$ and $i \in [F]$ and consider a situation where a source sequence $\boldsymbol{x}' \in \mathcal{S}^*$ is encoded with F starting from the code table f_i. Then the source sequence $\boldsymbol{x}'$ is encoded to the codeword sequence $f_i^*(\boldsymbol{x}')$, and the decoder reads it bit by bit from the beginning. Let $\boldsymbol{b} \preceq f_i^*(\boldsymbol{x}')$ be the sequence the decoder has read by a certain moment of the decoding process. If $\boldsymbol{b} = f_i^*(\boldsymbol{x})$ for some $\boldsymbol{x} \in \mathcal{S}^*$, then there are two possible cases, $\boldsymbol{x} \preceq \boldsymbol{x}'$ and $\boldsymbol{x} \npreceq \boldsymbol{x}'$. The k-bit delay decodability requires that it is always possible for the decoder to distinguish the two cases, $\boldsymbol{x} \preceq \boldsymbol{x}'$ and $\boldsymbol{x} \npreceq \boldsymbol{x}'$, by reading the following k bits $\boldsymbol{c} \in \mathcal{C}^k$ of the codeword sequence $f_i^*(\boldsymbol{x})$, that is, for any pair $(\boldsymbol{x}, \boldsymbol{c}) \in \mathcal{S}^* \times \mathcal{C}^k$, the decoder can distinguish the two cases, $\boldsymbol{x} \preceq \boldsymbol{x}'$ and $\boldsymbol{x} \npreceq \boldsymbol{x}'$ according to the pair $(\boldsymbol{x}, \boldsymbol{c})$. Thus, F is k-bit delay decodable if and only if for any pair $(\boldsymbol{x}, \boldsymbol{c}) \in \mathcal{S}^* \times \mathcal{C}^k$, it holds that $(\boldsymbol{x}, \boldsymbol{c})$ is f_i^**-positive* or f_i^**-negative* defined as follows.

Definition 2.2.4. *Let* $F(f,\tau) \in \mathscr{F}$ *and* $i \in [F]$.

(i) A pair $(\boldsymbol{x}, \boldsymbol{c}) \in \mathcal{S}^* \times \mathcal{C}^*$ *is said to be* f_i^*-positive *if for any* $\boldsymbol{x}' \in \mathcal{S}^*$*, if* $f_i^*(\boldsymbol{x})\boldsymbol{c} \preceq f_i^*(\boldsymbol{x}')$*, then* $\boldsymbol{x} \preceq \boldsymbol{x}'$.

(ii) A pair $(\boldsymbol{x}, \boldsymbol{c}) \in \mathcal{S}^* \times \mathcal{C}^*$ *is said to be* f_i^*-negative *if for any* $\boldsymbol{x}' \in \mathcal{S}^*$*, if* $f_i^*(\boldsymbol{x})\boldsymbol{c} \preceq f_i^*(\boldsymbol{x}')$*, then* $\boldsymbol{x} \npreceq \boldsymbol{x}'$.

Then the second definition of a k-bit delay decodable code-tuple is given as follows.

Definition 2.2.5. *Let* $k \geq 0$ *be an integer. A code-tuple* F *is said to be* k-bit delay decodable *if for any* $i \in [F]$ *and* $(\boldsymbol{x}, \boldsymbol{c}) \in \mathcal{S}^* \times \mathcal{C}^k$*, the pair* $(\boldsymbol{x}, \boldsymbol{c})$ *is* f_i^**-positive or* f_i^**-negative.*

Note that it is possible that a pair $(\boldsymbol{x}, \boldsymbol{c}) \in \mathcal{S}^* \times \mathcal{C}^*$ is f_i^*-positive and f_i^*-negative simultaneously. A pair $(\boldsymbol{x}, \boldsymbol{c}) \in \mathcal{S}^* \times \mathcal{C}^*$ is f_i^*-positive and f_i^*-negative simultaneously if and only if there is no sequence $\boldsymbol{x}'$ satisfying $f_i^*(\boldsymbol{x})\boldsymbol{c} \preceq f_i^*(\boldsymbol{x}')$.

The two definitions of a k-bit delay decodable code-tuple, Definitions 2.2.3 and 2.2.5, are indeed equivalent as shown in the following Lemma 2.2.4, which proof is deferred to Subsection 2.6.1.

Lemma 2.2.4. *For any* $F(f,\tau) \in \mathscr{F}$*, the following conditions (a) and (b) are equivalent.*

(a) For any $i \in [F]$ and $(\boldsymbol{x}, \boldsymbol{c}) \in \mathcal{S}^* \times \mathcal{C}^k$, the pair $(\boldsymbol{x}, \boldsymbol{c})$ is f_i^*-positive or f_i^*-negative.

(b) The code-tuple F satisfies Definition 2.2.3 (a) and (b).

The classes $\mathscr{F}_{k\text{-dec}}, k = 0, 1, 2, \ldots$ form a hierarchical structure $\mathscr{F}_{0\text{-dec}} \subseteq \mathscr{F}_{1\text{-dec}} \subseteq \mathscr{F}_{2\text{-dec}} \subseteq \cdots$. Namely, the following Lemma 2.2.5 holds.

Lemma 2.2.5. *For any two non-negative integers k, k', if $k \leq k'$, then $\mathscr{F}_{k\text{-dec}} \subseteq \mathscr{F}_{k'\text{-dec}}$.*

Proof of Lemma 2.2.5. Let $F(f, \tau) \in \mathscr{F}_{k\text{-dec}}$. Fix $i \in [F]$ and $(\boldsymbol{x}, \boldsymbol{c}') \in \mathcal{S}^* \times \mathcal{C}^{k'}$ arbitrarily. It suffices to prove that $(\boldsymbol{x}, \boldsymbol{c}')$ is f_i^*-positive or f_i^*-negative.

Let $\boldsymbol{c}$ be the prefix of $\boldsymbol{c}'$ of length k. Then for any $\boldsymbol{x}' \in \mathcal{S}^*$ such that $f_i^*(\boldsymbol{x})\boldsymbol{c}' \preceq f_i^*(\boldsymbol{x}')$, we have $f_i^*(\boldsymbol{x})\boldsymbol{c} \preceq f_i^*(\boldsymbol{x})\boldsymbol{c}' \preceq f_i^*(\boldsymbol{x}')$. Namely, $f_i^*(\boldsymbol{x})\boldsymbol{c}' \preceq f_i^*(\boldsymbol{x}')$ implies $f_i^*(\boldsymbol{x})\boldsymbol{c} \preceq f_i^*(\boldsymbol{x}')$. Hence, by Definition 2.2.4, if $(\boldsymbol{x}, \boldsymbol{c}')$ is f_i^*-positive (resp. f_i^*-negative), then also $(\boldsymbol{x}, \boldsymbol{c})$ is f_i^*-positive (resp. f_i^*-negative), respectively. Therefore, it follows that $F(f, \tau) \in \mathscr{F}_{k'\text{-dec}}$ from $F(f, \tau) \in \mathscr{F}_{k\text{-dec}}$. $\square$

The following Lemma 2.2.6 claims that a 0-bit delay decodable code-tuple (i.e., an instantaneous code) is always uniquely decodable (cf. Remark 2.2.1).

Lemma 2.2.6. *For any $F(f, \tau) \in \mathscr{F}_{0\text{-dec}}$ and $i \in [F]$, the following statements (i) and (ii) hold.*

(i) For any $\boldsymbol{x} \in \mathcal{S}^*$, the pair $(\boldsymbol{x}, \lambda)$ is f_i^*-positive.

(ii) f_i^* is injective.

Proof of Lemma 2.2.6. (Proof of (i)): By $F \in \mathscr{F}_{0\text{-dec}}$, the pair $(\boldsymbol{x}, \lambda)$ is f_i^*-positive or f_i^*-negative. However, since $f_i^*(\boldsymbol{x}) \preceq f_i^*(\boldsymbol{x})$ and $\boldsymbol{x} \preceq \boldsymbol{x}$, the pair $(\boldsymbol{x}, \lambda)$ must be f_i^*-positive.

(Proof of (ii)): By (i) of this lemma, we have

$$^{\forall}\boldsymbol{x}, \boldsymbol{x}' \in \mathcal{S}^*; (f_i^*(\boldsymbol{x}) \preceq f_i^*(\boldsymbol{x}') \implies \boldsymbol{x} \preceq \boldsymbol{x}'). \tag{2.40}$$

Choose $\boldsymbol{y}, \boldsymbol{y}' \in \mathcal{S}^*$ such $f_i^*(\boldsymbol{y}) = f_i^*(\boldsymbol{y}')$ arbitrarily. Then we have $f_i^*(\boldsymbol{y}) \preceq f_i^*(\boldsymbol{y}')$ and $f_i^*(\boldsymbol{y}') \preceq f_i^*(\boldsymbol{y})$. Thus, by (2.40), we obtain $\boldsymbol{y} \preceq \boldsymbol{y}'$ and $\boldsymbol{y}' \preceq \boldsymbol{y}$, that is, $\boldsymbol{y} = \boldsymbol{y}'$. Consequently, f_i^* is injective. $\square$

A 0-bit delay decodable code-tuple is also characterized as a code-tuple all of which code tables are *prefix-free* as below.

Definition 2.2.6. *For $F(f, \tau) \in \mathscr{F}$ and $i \in [F]$, the mapping f_i is said to be* prefix-free *if for any $s, s' \in \mathcal{S}$, if $f_i(s) \preceq f_i(s')$, then $s = s'$.*

Lemma 2.2.7. *A code-tuple $F(f, \tau)$ satisfies $F \in \mathscr{F}_{0\text{-dec}}$ if and only if for any $i \in [F]$, the mapping f_i is prefix-free.*

Proof of Lemma 2.2.7. (Necessity) Assume $F \in \mathscr{F}_{0\text{-dec}}$ and choose $i \in [F]$ arbitrarily. By Lemma 2.2.6 (i), the pair $(\boldsymbol{x}, \lambda)$ is f_i^*-positive. Thus, (2.40) holds. In particular, we have

$$^{\forall}s, s' \in \mathcal{S}; (f_i^*(s) \preceq f_i^*(s') \implies s \preceq s'). \tag{2.41}$$

Since $s \preceq s'$ implies $s = s'$, the mapping f_i is prefix-free.

(Sufficiency) Assume that for any $i \in [F]$, the mapping f_i is prefix-free. To prove $F \in \mathscr{F}_{0\text{-dec}}$, it suffices to prove (2.40) for arbitrarily fixed $i \in [F]$. We prove it by induction for $|\boldsymbol{x}|$.

For the base case $|\boldsymbol{x}| = 0$, clearly we have $\boldsymbol{x} \preceq \boldsymbol{x}'$ for any $\boldsymbol{x}' \in \mathcal{S}^*$.

Let $l \geq 1$ and assume that (2.40) is true for the case $|\boldsymbol{x}| < l$ as the induction hypothesis. We prove (2.40) for the case $|\boldsymbol{x}| = l$. Choose $\boldsymbol{x}' \in \mathcal{S}^*$ such that $f_i^*(\boldsymbol{x}) \preceq f_i^*(\boldsymbol{x}')$ arbitrarily. Then by (2.4), we have

$$f_i(x_1) f^*_{\tau_i^*(x_1)}(\mathrm{suff}(\boldsymbol{x})) \preceq f_i(x'_1) f^*_{\tau_i^*(x'_1)}(\mathrm{suff}(\boldsymbol{x}')). \tag{2.42}$$

Thus, $f_i(x_1) \preceq f_i(x'_1)$ or $f_i(x_1) \succeq f_i(x'_1)$ holds. Hence, since f_i is prefix-free, we obtain

$$x_1 = x'_1. \tag{2.43}$$

By (2.42) and (2.43), we have $f_i(x_1) f^*_{\tau_i^*(x_1)}(\mathrm{suff}(\boldsymbol{x})) \preceq f_i(x_1) f^*_{\tau_i^*(x'_1)}(\mathrm{suff}(\boldsymbol{x}'))$. Thus, we have $f^*_{\tau_i^*(x_1)}(\mathrm{suff}(\boldsymbol{x})) \preceq f^*_{\tau_i^*(x'_1)}(\mathrm{suff}(\boldsymbol{x}'))$. By the induction hypothesis,

$$\mathrm{suff}(\boldsymbol{x}) \preceq \mathrm{suff}(\boldsymbol{x}'). \tag{2.44}$$

By (2.43) and (2.44), we obtain $\boldsymbol{x} \preceq \boldsymbol{x}'$. □

2.3 Extendable Code-tuples

For the code-tuple $F^{(\alpha)}$ in Table 2.1, we can see that $f_2^{(\alpha)^*}(\boldsymbol{x}) = \lambda$ for any $\boldsymbol{x} \in \mathcal{S}^*$. To exclude such abnormal and useless code-tuples, we introduce a class $\mathscr{F}_{\mathrm{ext}}$ in the following Definition 2.3.1.

Definition 2.3.1. *A code-tuple F is said to be* extendable *if $\mathcal{P}^1_{F,i} \neq \emptyset$ for any $i \in [F]$. We define $\mathscr{F}_{\mathrm{ext}}$ as the set of all extendable code-tuples, that is,*

$$\mathscr{F}_{\mathrm{ext}} \coloneqq \{F \in \mathscr{F} : {}^{\forall} i \in [F]; \mathcal{P}^1_{F,i} \neq \emptyset\}. \tag{2.45}$$

Example 2.3.1. *The code-tuple $F^{(\alpha)}$ in Table 2.1 is not extendable because $\mathcal{P}^1_{F^{(\alpha)},2} = \emptyset$ by Table 2.2. The code-tuples $F^{(\beta)}$ and $F^{(\gamma)}$ in Table 2.1 are extendable.*

The following Lemma 2.3.1 shows that for an extendable code-tuple $F(f, \tau)$, we can extend the length of $f^*_i(\boldsymbol{x})$ as long as we want by appending symbols to $\boldsymbol{x}$ appropriately.

Lemma 2.3.1. *A code-tuple $F(f, \tau)$ is extendable if and only if for any $i \in [F]$ and integer $l \geq 0$, there exists $\boldsymbol{x} \in \mathcal{S}^*$ such that $|f^*_i(\boldsymbol{x})| \geq l$.*

Proof of Lemma 2.3.1. (Sufficiency) Fix $i \in [F]$ arbitrarily. Applying the assumption with $l = 1$, we see that there exists $\boldsymbol{x} \in \mathcal{S}^*$ such that $|f^*_i(\boldsymbol{x})| \geq 1$. Then there exists $c \in \mathcal{C}$ such that $f^*_i(\boldsymbol{x}) \succeq c$, which leads to $c \in \mathcal{P}^1_{F,i}$ by (2.16), that is, $\mathcal{P}^1_{F,i} \neq \emptyset$ as desired.

(Necessity) Assume $F \in \mathscr{F}_{\mathrm{ext}}$. We prove by induction for l. The base case $l = 0$ is trivial. We consider the induction step for $l \geq 1$. By the induction hypothesis, there exists $\boldsymbol{x} \in \mathcal{S}^*$ such that

$$|f^*_i(\boldsymbol{x})| \geq l - 1. \tag{2.46}$$

Also, by $F \in \mathscr{F}_{\mathrm{ext}}$, there exists $c \in \mathcal{P}^1_{F,\tau^*_i(\boldsymbol{x})}$. By (2.16), there exists $\boldsymbol{y} \in \mathcal{S}^*$ such that

$$f^*_{\tau^*_i(\boldsymbol{x})}(\boldsymbol{y}) \succeq c. \tag{2.47}$$

Thus, we obtain

$$|f^*_i(\boldsymbol{xy})| \overset{\text{(A)}}{=} |f^*_i(\boldsymbol{x})| + |f^*_{\tau^*_i(\boldsymbol{x})}(\boldsymbol{y})| \overset{\text{(B)}}{\geq} (l-1) + 1 = l, \tag{2.48}$$

where (A) follows from Lemma 2.1.1 (i), and (B) follows from (2.46) and (2.47). This completes the induction. □

This property yields the following Lemma 2.3.2 and Corollary 2.3.1.

Lemma 2.3.2. *Let k, k' be two integers such that $0 \leq k \leq k'$. For any $F(f, \tau) \in \mathscr{F}_{\mathrm{ext}}, i \in [F]$, $\boldsymbol{b} \in \mathcal{C}^*$, and $\boldsymbol{c} \in \mathcal{C}^k$, the following statements (i) and (ii) hold.*

(i) $\boldsymbol{c} \in \mathcal{P}^{k}_{F,i}(\boldsymbol{b}) \iff {}^{\exists}\boldsymbol{c}' \in \mathcal{C}^{k'-k}; \boldsymbol{cc}' \in \mathcal{P}^{k'}_{F,i}(\boldsymbol{b})$.

(ii) $\boldsymbol{c} \in \bar{\mathcal{P}}^{k}_{F,i}(\boldsymbol{b}) \iff {}^{\exists}\boldsymbol{c}' \in \mathcal{C}^{k'-k}; \boldsymbol{cc}' \in \bar{\mathcal{P}}^{k'}_{F,i}(\boldsymbol{b})$.

Proof of Lemma 2.3.2. We prove (i) only because (ii) follows by the similar argument.

($\Longrightarrow$) Assume $\boldsymbol{c} \in \mathcal{P}^{k}_{F,i}(\boldsymbol{b})$. Then by (2.12), there exists $\boldsymbol{x} \in \mathcal{S}^{+}$ such that

$$f^{*}_{i}(\boldsymbol{x}) \succeq \boldsymbol{bc}, \tag{2.49}$$

$$f_{i}(x_1) \succeq \boldsymbol{b}. \tag{2.50}$$

By $F \in \mathscr{F}_{\mathrm{ext}}$ and Lemma 2.3.1, there exists $\boldsymbol{y} \in \mathcal{S}^{*}$ such that

$$|f^{*}_{\tau^{*}_{i}(\boldsymbol{x})}(\boldsymbol{y})| \geq k' - k. \tag{2.51}$$

Hence, we have

$$|f^{*}_{i}(\boldsymbol{xy})| \overset{\text{(A)}}{=} |f^{*}_{i}(\boldsymbol{x})| + |f^{*}_{\tau^{*}_{i}(\boldsymbol{x})}(\boldsymbol{y})| \overset{\text{(B)}}{\geq} |\boldsymbol{bc}| + k' - k, \tag{2.52}$$

where (A) follows from Lemma 2.1.1 (i), and (B) follows from (2.49) and (2.51). By (2.49) and (2.52), there exists $\boldsymbol{c}' \in \mathcal{C}^{k'-k}$ such that

$$f^{*}_{i}(\boldsymbol{xy}) \succeq \boldsymbol{bcc}'. \tag{2.53}$$

Equations (2.50) and (2.53) lead to $\boldsymbol{cc}' \in \mathcal{P}^{k'}_{F,i}(\boldsymbol{b})$ by (2.12).

($\Longleftarrow$) Assume that there exists $\boldsymbol{c}' \in \mathcal{C}^{k'-k}$ such that $\boldsymbol{cc}' \in \mathcal{P}^{k'}_{F,i}(\boldsymbol{b})$. Then by (2.12), there exists $\boldsymbol{x} \in \mathcal{S}^{+}$ such that $f^{*}_{i}(\boldsymbol{x}) \succeq \boldsymbol{bcc}'$ and $f_{i}(x_1) \succeq \boldsymbol{b}$. This clearly implies $f^{*}_{i}(\boldsymbol{x}) \succeq \boldsymbol{bc}$ and $f_{i}(x_1) \succeq \boldsymbol{b}$, which leads to $\boldsymbol{c} \in \mathcal{P}^{k}_{F,i}(\boldsymbol{b})$ by (2.12). □

Corollary 2.3.1. *For any $F \in \mathscr{F}_{\mathrm{ext}}$, $i \in [F]$, and $\boldsymbol{b} \in \mathcal{C}^{*}$, the following statements (i) and (ii) hold.*

(i) *(a)* *For any integer $k \geq 0$, the following equivalence holds: $\mathcal{P}^{k}_{F,i}(\boldsymbol{b}) = \emptyset \iff \mathcal{P}^{0}_{F,i}(\boldsymbol{b}) = \emptyset$.*

(b) *For any integers k and k' such that $0 \leq k \leq k'$, we have $|\mathcal{P}^{k}_{F,i}(\boldsymbol{b})| \leq |\mathcal{P}^{k'}_{F,i}(\boldsymbol{b})|$.*

(ii) *(a)* *For any integer $k \geq 0$, the following equivalence holds: $\bar{\mathcal{P}}^{k}_{F,i}(\boldsymbol{b}) = \emptyset \iff \bar{\mathcal{P}}^{0}_{F,i}(\boldsymbol{b}) = \emptyset$.*

(b) *For any integers k and k' such that $0 \le k \le k'$, we have $|\bar{\mathcal{P}}^{k}_{F,i}(\boldsymbol{b})| \le |\bar{\mathcal{P}}^{k'}_{F,i}(\boldsymbol{b})|$.*

Also, the following Lemma 2.3.3 gives a lower bound of the length of a codeword sequence for $F \in \mathscr{F}_{\mathrm{ext}} \cap \mathscr{F}_{k\text{-dec}}$. See Subsection 2.6.2 for the proof of Lemma 2.3.3.

Lemma 2.3.3. *For any integer $k \ge 0$, $F(f, \tau) \in \mathscr{F}_{\mathrm{ext}} \cap \mathscr{F}_{k\text{-dec}}, i \in [F]$, and $\boldsymbol{x} \in \mathcal{S}^*$, we have $|f^*_i(\boldsymbol{x})| \ge \lfloor |\boldsymbol{x}|/|F| \rfloor$.*

2.4 Average Codeword Length of Code-Tuple

We introduce the average codeword length $L(F)$ of a code-tuple F. From now on, we fix an arbitrary probability distribution μ of the source symbols, that is, a real-valued function $\mu : \mathcal{S} \to \mathbb{R}$ such that $\sum_{s \in \mathcal{S}} \mu(s) = 1$ and $0 < \mu(s) \le 1$ for any $s \in \mathcal{S}$. Note that we exclude the case where $\mu(s) = 0$ for some $s \in \mathcal{S}$ without loss of generality.

First, for $F(f, \tau) \in \mathscr{F}$ and $i, j \in [F]$, we define the transition probability $Q_{i,j}(F)$ as the probability of using the code table f_j next after using the code table f_i in the encoding process.

Definition 2.4.1. *For $F(f, \tau) \in \mathscr{F}$ and $i, j \in [F]$, we define the* transition probability $Q_{i,j}(F)$ *as*

$$Q_{i,j}(F) \coloneqq \sum_{s \in \mathcal{S}, \tau_i(s) = j} \mu(s). \tag{2.54}$$

We also define the transition probability matrix $Q(F)$ *as the following $|F| \times |F|$ matrix:*

$$\begin{bmatrix} Q_{0,0}(F) & Q_{0,1}(F) & \cdots & Q_{0,|F|-1}(F) \\ Q_{1,0}(F) & Q_{1,1}(F) & \cdots & Q_{1,|F|-1}(F) \\ \vdots & \vdots & \ddots & \vdots \\ Q_{|F|-1,0}(F) & Q_{|F|-1,1}(F) & \cdots & Q_{|F|-1,|F|-1}(F) \end{bmatrix}. \tag{2.55}$$

We fix $F \in \mathscr{F}$ and consider the encoding process with F. Let $I_i \in [F]$ be the index of the code table used to encode the i-th symbol of a source sequence for $i = 1, 2, 3, \ldots$. Then $\{I_i\}_{i=1,2,3,\ldots}$ is a Markov process with the transition probability matrix $Q(F)$. We consider a stationary distribution of the Markov process $\{I_i\}_{i=1,2,3,\ldots}$, formally defined as follows.

Definition 2.4.2. *For $F \in \mathscr{F}$, a solution $\boldsymbol{\pi} = (\pi_0, \pi_1, \ldots, \pi_{|F|-1}) \in \mathbb{R}^{|F|}$ of the following simultaneous equations (2.56) and (2.57) is called a* stationary distribution *of F:*

$$\begin{cases} \boldsymbol{\pi} Q(F) = \boldsymbol{\pi}, & (2.56) \\ \sum_{i \in [F]} \pi_i = 1. & (2.57) \end{cases}$$

A code-tuple has at least one stationary distribution without a negative element as shown in the following Lemma 2.4.1. See Subsection 2.6.3 for the proof of Lemma 2.4.1.

Lemma 2.4.1. *For any $F \in \mathscr{F}$, there exists a stationary distribution $\boldsymbol{\pi} = (\pi_0, \pi_1, \ldots, \pi_{|F|-1})$ of F such that $\pi_i \geq 0$ for any $i \in [F]$.*

As stated later in Definition 2.4.4, the average codeword length $L(F)$ of F is defined depending on the stationary distribution $\boldsymbol{\pi}$ of F. However, it is possible that a code-tuple has multiple stationary distributions. Therefore, we limit the scope of consideration to a class $\mathscr{F}_{\mathrm{reg}}$ defined as the following Definition 2.4.3, which is the class of code-tuples with a unique stationary distribution.

Definition 2.4.3. *A code-tuple F is said to be* regular *if F has a unique stationary distribution. We define $\mathscr{F}_{\mathrm{reg}}$ as the set of all regular code-tuples, that is,*

$$\mathscr{F}_{\mathrm{reg}} := \{F \in \mathscr{F} : F \text{ is regular}\}. \tag{2.58}$$

For $F \in \mathscr{F}_{\mathrm{reg}}$, we define $\boldsymbol{\pi}(F) = (\pi_0(F), \pi_1(F), \ldots, \pi_{|F|-1}(F))$ as the unique stationary distribution of F.

Since the transition probability matrix $Q(F)$ depends on μ, it might seem that the class $\mathscr{F}_{\mathrm{reg}}$ also depends on μ. However, we show later as Lemma 2.5.2 that in fact $\mathscr{F}_{\mathrm{reg}}$ is independent of μ. More precisely, whether a code-tuple $F(f, \tau)$ belongs to $\mathscr{F}_{\mathrm{reg}}$ depends only on $\tau_0, \tau_1, \ldots, \tau_{|F|-1}$. We also note that for any $F \in \mathscr{F}_{\mathrm{reg}}$, the unique stationary distribution $\boldsymbol{\pi}(F)$ of F satisfies $\pi_i(F) \geq 0$ for any $i \in [F]$ by Lemma 2.4.1.

The asymptotical performance (i.e., average codeword length per symbol) of a regular code-tuple does not depend on which code table we start encoding: the average codeword length $L(F)$ of a regular code-tuple F is the weighted sum of the average codeword lengths of the code tables $f_0, f_1, \ldots, f_{|F|-1}$ weighted by the stationary distribution $\boldsymbol{\pi}(F)$. Namely, $L(F)$ is defined as the following Definition 2.4.4.

Definition 2.4.4. *For $F(f,\tau) \in \mathscr{F}$ and $i \in [F]$, we define the* average codeword length $L_i(F)$ of the single code table $f_i : \mathcal{S} \to \mathcal{C}^*$ *as*

$$L_i(F) \coloneqq \sum_{s \in \mathcal{S}} |f_i(s)| \cdot \mu(s). \tag{2.59}$$

For $F \in \mathscr{F}_{\mathrm{reg}}$, we define the average codeword length $L(F)$ of the code-tuple F *as*

$$L(F) \coloneqq \sum_{i \in [F]} \pi_i(F) L_i(F). \tag{2.60}$$

Example 2.4.1. *We consider $F \coloneqq F^{(\gamma)}$ of Table 2.1, where $(\mu(\mathrm{a}), \mu(\mathrm{b}), \mu(\mathrm{c}), \mu(\mathrm{d})) = (0.1, 0.2, 0.3, 0.4)$. We have*

$$Q(F) = \begin{bmatrix} 0.4 & 0.2 & 0.4 \\ 0.2 & 0.4 & 0.4 \\ 0.2 & 0.1 & 0.7 \end{bmatrix}. \tag{2.61}$$

The simultaneous equations (2.56) and (2.57) have the unique solution $\boldsymbol{\pi}(F) = (\pi_0(F), \pi_1(F), \pi_2(F)) = (1/4, 5/28, 4/7)$. Hence, we have $F \in \mathscr{F}_{\mathrm{reg}}$. Also, we have

$$L_0(F) = 2.6, \quad L_1(F) = 3.7, \quad L_2(F) = 4.2. \tag{2.62}$$

Therefore, the average codeword length $L(F)$ of the code-tuple F is given as

$$L(F) = \pi_0(F)L_0(F) + \pi_1(F)L_1(F) + \pi_2(F)L_2(F) \approx 3.7107. \tag{2.63}$$

Remark 2.4.1. *Note that $Q(F), L_i(F), L(F)$, and $\boldsymbol{\pi}(F)$ depend on μ. However, since we are now discussing on a fixed μ, the average codeword length $L_i(F)$ of f_i (resp. the transition probability matrix $Q(F)$) is determined only by the mapping f_i (resp. $\tau_0, \tau_1, \ldots, \tau_{|F|-1}$) and therefore the stationary distribution $\boldsymbol{\pi}(F)$ of a regular code-tuple F is also determined only by $\tau_0, \tau_1, \ldots, \tau_{|F|-1}$.*

2.5 Irreducible Code-tuples and Irreducible Parts

As we can see from (2.60), the code tables f_i of $F(f,\tau) \in \mathscr{F}_{\mathrm{reg}}$ such that $\pi_i(F) = 0$ does not contribute to $L(F)$. It is useful to remove such non-essential code tables and obtain an *irreducible* code-tuple: we say that a regular code-tuple F is *irreducible* if $\pi_i(F) > 0$ for any $i \in [F]$ as formally defined later in Definition 2.5.3. In this section, we introduce an *irreducible part* of $F \in \mathscr{F}_{\mathrm{reg}}$, which is an irreducible code-tuple obtained by removing all the code tables f_i such that $\pi_i(F) = 0$ from F. The formal definition of an irreducible part of F is stated using a notion of *homomorphism* defined in the following Definition 2.5.1.

Definition 2.5.1. *For $F(f, \tau), F'(f', \tau') \in \mathscr{F}$, a mapping $\varphi : [F'] \to [F]$ is called a* homomorphism from F' to F *if*

$$f'_i(s) = f_{\varphi(i)}(s), \tag{2.64}$$

$$\varphi(\tau'_i(s)) = \tau_{\varphi(i)}(s) \tag{2.65}$$

for any $i \in [F']$ and $s \in \mathcal{S}$.

Given a homomorphism of code-tuples, the following Lemma 2.5.1 holds between the two code-tuples. See Appendix 2.6.4 for the proof of Lemma 2.5.1.

Lemma 2.5.1. *For any $F(f, \tau), F'(f', \tau') \in \mathscr{F}$ and a homomorphism $\varphi : [F'] \to [F]$ from F' to F , the following statements (i)–(vi) hold.*

(i) For any $i \in [F']$ and $\boldsymbol{x} \in \mathcal{S}^*$, we have $f'^*_i(\boldsymbol{x}) = f^*_{\varphi(i)}(\boldsymbol{x})$ and $\varphi(\tau'^*_i(\boldsymbol{x})) = \tau^*_{\varphi(i)}(\boldsymbol{x})$.

(ii) For any $i \in [F']$ and $\boldsymbol{b} \in \mathcal{C}^*$, we have $\mathcal{P}^*_{F',i}(\boldsymbol{b}) = \mathcal{P}^*_{F,\varphi(i)}(\boldsymbol{b})$ and $\bar{\mathcal{P}}^*_{F',i}(\boldsymbol{b}) = \bar{\mathcal{P}}^*_{F,\varphi(i)}(\boldsymbol{b})$.

(iii) For any stationary distribution $\boldsymbol{\pi}' = (\pi'_0, \pi'_1, \ldots, \pi'_{|F'|-1})$ of F', the vector $\boldsymbol{\pi} = (\pi_0, \pi_1, \ldots, \pi_{|F|-1}) \in \mathbb{R}^{|F|}$ defined as

$$\pi_j = \sum_{j' \in \mathcal{A}_j} \pi'_{j'} \text{ for } j \in [F] \tag{2.66}$$

is a stationary distribution of F, where

$$\mathcal{A}_i \coloneqq \{i' \in [F'] : \varphi(i') = i\} \tag{2.67}$$

for $i \in [F]$.

(iv) If $F \in \mathscr{F}_{\mathrm{ext}}$, then $F' \in \mathscr{F}_{\mathrm{ext}}$.

(v) If $F, F' \in \mathscr{F}_{\mathrm{reg}}$, then $L(F') = L(F)$.

(vi) For any integer $k \geq 0$, if $F \in \mathscr{F}_{k\text{-dec}}$, then $F' \in \mathscr{F}_{k\text{-dec}}$.

We also introduce the set $\mathcal{R}_F$ for $F \in \mathscr{F}$ as the following Definition 2.5.2. We state in Lemma 2.5.2 that we can characterize a regular code-tuple F by $\mathcal{R}_F$.

Definition 2.5.2. *For $F(f, \tau) \in \mathscr{F}$, we define $\mathcal{R}_F$ as*

$$\mathcal{R}_F \coloneqq \{i \in [F] : {}^{\forall} j \in [F]; {}^{\exists}\boldsymbol{x} \in \mathcal{S}^*; \tau_j^*(\boldsymbol{x}) = i\}. \tag{2.68}$$

Namely, $\mathcal{R}_F$ is the set of indices i of the code tables such that for any $j \in [F]$, there exists $\boldsymbol{x} \in \mathcal{S}^*$ such that $\tau_j^*(\boldsymbol{x}) = i$.

Example 2.5.1. *First, we consider $F(f, \tau) \coloneqq F^{(\alpha)}$ in Table 2.1. Then we confirm $\mathcal{R}_F = \{2\}$ as follows.*

- $0 \notin \mathcal{R}_F$ *because there exists no $\boldsymbol{x} \in \mathcal{S}^*$ such that $\tau_2^*(\boldsymbol{x}) = 0$.*
- $1 \notin \mathcal{R}_F$ *because there exists no $\boldsymbol{x} \in \mathcal{S}^*$ such that $\tau_2^*(\boldsymbol{x}) = 1$.*
- $2 \in \mathcal{R}_F$ *because $\tau_0^*(\mathrm{bc}) = \tau_1^*(\mathrm{c}) = \tau_2^*(\lambda) = 2$.*

Next, we consider $F(f, \tau) \coloneqq F^{(\beta)}$ in Table 2.1. Then we confirm $\mathcal{R}_F = \emptyset$ as follows.

- $0 \notin \mathcal{R}_F$ *because there exists no $\boldsymbol{x} \in \mathcal{S}^*$ such that $\tau_1^*(\boldsymbol{x}) = 0$.*
- $1 \notin \mathcal{R}_F$ *because there exists no $\boldsymbol{x} \in \mathcal{S}^*$ such that $\tau_2^*(\boldsymbol{x}) = 1$.*
- $2 \notin \mathcal{R}_F$ *because there exists no $\boldsymbol{x} \in \mathcal{S}^*$ such that $\tau_1^*(\boldsymbol{x}) = 2$.*

Lastly, we consider $F(f, \tau) \coloneqq F^{(\gamma)}$ in Table 2.1. Then we confirm $\mathcal{R}_F = \{0, 1, 2\}$ as follows.

- $0 \in \mathcal{R}_F$ *because $\tau_0^*(\lambda) = \tau_1^*(\mathrm{b}) = \tau_2^*(\mathrm{b}) = 0$.*
- $1 \in \mathcal{R}_F$ *because $\tau_0^*(\mathrm{b}) = \tau_1^*(\lambda) = \tau_2^*(\mathrm{a}) = 1$.*
- $2 \in \mathcal{R}_F$ *because $\tau_0^*(\mathrm{d}) = \tau_1^*(\mathrm{d}) = \tau_2^*(\lambda) = 2$.*

Lemma 2.5.2. *For any $F \in \mathscr{F}$, the following statements (i) and (ii) hold.*

(i) $F \in \mathscr{F}_{\mathrm{reg}}$ *if and only if* $\mathcal{R}_F \neq \emptyset$.

(ii) If $F \in \mathscr{F}_{\mathrm{reg}}$, *then for any* $i \in [F]$, *the following equivalence relation holds:* $\pi_i(F) > 0 \iff i \in \mathcal{R}_F$.

The proof of Lemma 2.5.2 is given in Subsection 2.6.5.

Since $\mathcal{R}_F$ does not depend on μ, we can see from Lemma 2.5.2 (i) that the class $\mathscr{F}_{\mathrm{reg}}$ is determined independently of μ as mentioned before.

By Lemma 2.5.2 (ii), a regular code-tuple $F(f, \tau)$ satisfies $\pi_i(F) > 0$ for any $i \in [F]$ if and only if F is an irreducible code-tuple defined as follows.

Definition 2.5.3. *A code-tuple* F *is said to be* irreducible *if* $\mathcal{R}_F = [F]$. *We define* $\mathscr{F}_{\mathrm{irr}}$ *as the set of all irreducible code-tuples, that is,* $\mathscr{F}_{\mathrm{irr}} := \{F \in \mathscr{F} : \mathcal{R}_F = [F]\}$.

Note that $\mathscr{F}_{\mathrm{irr}} \subseteq \mathscr{F}_{\mathrm{reg}}$ since $F \in \mathscr{F}_{\mathrm{reg}}$ is equivalent to $\mathcal{R}_F \neq \emptyset$ by Lemma 2.5.2 (i).

Now we define an *irreducible part* $\bar{F}$ of a code-tuple F as the following Definition 2.5.4.

Definition 2.5.4. *An irreducible code-tuple* $\bar{F}$ *is called an* irreducible part *of a code-tuple* F *if there exists an injective homomorphism* $\varphi : [\bar{F}] \to [F]$ *from* $\bar{F}$ *to* F.

The following property of $\bar{F}$ is immediately from Definition 2.5.4 and Lemma 2.5.1 (iv)–(vi).

Lemma 2.5.3. *For any integer* $k \geq 0$, $F \in \mathscr{F}_{\mathrm{reg}} \cap \mathscr{F}_{\mathrm{ext}} \cap \mathscr{F}_{k\text{-dec}}$, *and an irreducible part* $\bar{F}$ *of* F, *we have* $\bar{F} \in \mathscr{F}_{\mathrm{irr}} \cap \mathscr{F}_{\mathrm{ext}} \cap \mathscr{F}_{k\text{-dec}}$ *and* $L(\bar{F}) = L(F)$.

The existence of an irreducible part is guaranteed as the following Lemma 2.5.4. See Appendix 2.6.6 for the proof of Lemma 2.5.4.

Lemma 2.5.4. *For any* $F \in \mathscr{F}_{\mathrm{reg}}$, *there exists an irreducible part* $\bar{F}$ *of* F.

2.6 Proofs of Lemmas in Chapter 2

2.6.1 Proof of Lemma 2.2.4

Proof of Lemma 2.2.4. ((a) $\implies$ (b)): We show the contraposition. Assume that (b) does not hold. We consider the following two cases separately: the case where Definition 2.2.3 (a) is false and the case where Definition 2.2.3 (b) is false.

- The case where Definition 2.2.3 (a) is false: Then there exist $i \in [F]$, $s \in \mathcal{S}$, and $\boldsymbol{c} \in \mathcal{P}^k_{F,\tau_i(s)} \cap \bar{\mathcal{P}}^k_{F,i}(f_i(s))$. By (2.13) and (2.16), there exist $\boldsymbol{x} \in \mathcal{S}^*$ and $\boldsymbol{x}' \in \mathcal{S}^+$ such that

$$f^*_{\tau_i(s)}(\boldsymbol{x}) \succeq \boldsymbol{c}, \tag{2.69}$$

$$f^*_i(\boldsymbol{x}') \succeq f_i(s)\boldsymbol{c}, \tag{2.70}$$

$$f_i(x'_1) \succ f_i(s). \tag{2.71}$$

  We have

$$f^*_i(s\boldsymbol{x}) \overset{\text{(A)}}{=} f_i(s) f^*_{\tau_i(s)}(\boldsymbol{x}) \overset{\text{(B)}}{\succeq} f_i(s)\boldsymbol{c}, \tag{2.72}$$

  where (A) follows from (2.4), and (B) follows from (2.69). By (2.72) and $s \preceq s\boldsymbol{x}$, the pair $(s, \boldsymbol{c})$ is not f^*_i-negative. On the other hand, since $s \neq x'_1$ by (2.71), we have $s \not\preceq \boldsymbol{x}'$. Hence, by (2.70), the pair $(s, \boldsymbol{c})$ is not f^*_i-positive. Since the pair $(s, \boldsymbol{c})$ is neither f^*_i-positive nor f^*_i-negative, the condition (a) does not hold.

- The case where Definition 2.2.3 (b) is false: Then there exist $i \in [F]$, $s, s' \in \mathcal{S}$, and $\boldsymbol{c} \in \mathcal{P}^k_{F,\tau_i(s)} \cap \mathcal{P}^k_{F,\tau_i(s')}$ such that $s \neq s'$ and

$$f_i(s) = f_i(s'). \tag{2.73}$$

  By (2.16), there exist $\boldsymbol{x}, \boldsymbol{x}' \in \mathcal{S}^*$ such that

$$f^*_{\tau_i(s)}(\boldsymbol{x}) \succeq \boldsymbol{c}, \tag{2.74}$$

$$f^*_{\tau_i(s')}(\boldsymbol{x}') \succeq \boldsymbol{c}. \tag{2.75}$$

  Thus, we have

$$f^*_i(s\boldsymbol{x}) \overset{\text{(A)}}{=} f_i(s) f^*_{\tau_i(s)}(\boldsymbol{x}) \overset{\text{(B)}}{\succeq} f_i(s)\boldsymbol{c}, \tag{2.76}$$

$$f^*_i(s'\boldsymbol{x}') \overset{\text{(C)}}{=} f_i(s') f^*_{\tau_i(s')}(\boldsymbol{x}') \overset{\text{(D)}}{=} f_i(s) f^*_{\tau_i(s')}(\boldsymbol{x}') \overset{\text{(E)}}{\succeq} f_i(s)\boldsymbol{c}, \tag{2.77}$$

  where (A) follows from (2.4), (B) follows from (2.74), (C) follows from (2.4), (D) follows from (2.73), and (E) follows from (2.75). By (2.76) and $s \preceq s\boldsymbol{x}$, the pair $(s, \boldsymbol{c})$ is not f^*_i-negative. On the other hand, by $s \not\preceq s'\boldsymbol{x}$ and (2.77), the pair $(s, \boldsymbol{c})$ is not f^*_i-positive. Since the pair $(s, \boldsymbol{c})$ is neither f^*_i-positive nor f^*_i-negative, the condition (a) does not hold.

((b) $\implies$ (a)): We show the contraposition. Assume that (a) does not hold. Then there exist $i \in [F]$ and $(\boldsymbol{x}, \boldsymbol{c}) \in \mathcal{S}^* \times \mathcal{C}^k$ such that $(\boldsymbol{x}, \boldsymbol{c})$ is neither f_i^*-positive nor f_i^*-negative. Thus, there exist $\boldsymbol{x}', \boldsymbol{x}'' \in \mathcal{S}^*$ such that

$$f_i^*(\boldsymbol{x})\boldsymbol{c} \preceq f_i^*(\boldsymbol{x}'), \tag{2.78}$$

$$f_i^*(\boldsymbol{x})\boldsymbol{c} \preceq f_i^*(\boldsymbol{x}''), \tag{2.79}$$

$$\boldsymbol{x} \preceq \boldsymbol{x}', \tag{2.80}$$

$$\boldsymbol{x} \not\preceq \boldsymbol{x}''. \tag{2.81}$$

We consider the following two cases separately: the case $\boldsymbol{x} \succeq \boldsymbol{x}''$ and the case $\boldsymbol{x} \not\succeq \boldsymbol{x}''$.

- The case $\boldsymbol{x} \succeq \boldsymbol{x}''$: By Lemma 2.1.1 (iii), we have

  $$f_i^*(\boldsymbol{x}) \succeq f_i^*(\boldsymbol{x}''). \tag{2.82}$$

  Hence, by (2.79), it must hold that $\boldsymbol{c} = \lambda$. Namely, only $k = 0$ is possible now.

  Since (2.81) and $\boldsymbol{x} \succeq \boldsymbol{x}''$ lead to $\boldsymbol{x} \succ \boldsymbol{x}''$, there exists $\boldsymbol{u} \in \mathcal{S}^+$ such that $\boldsymbol{x} = \boldsymbol{x}''\boldsymbol{u}$. Defining $j := \tau_i^*(\boldsymbol{x}'')$, we have

  $$f_i^*(\boldsymbol{x}) \overset{\text{(A)}}{=} f_i^*(\boldsymbol{x}'')f_j^*(\boldsymbol{u}) \overset{\text{(B)}}{=} f_i^*(\boldsymbol{x})f_j^*(\boldsymbol{u}) \overset{\text{(C)}}{=} f_i^*(\boldsymbol{x})f_j(u_1)f_{\tau_j(u_1)}^*(\mathrm{suff}(\boldsymbol{u})), \tag{2.83}$$

  where (A) follows from Lemma 2.1.1 (i), (B) follows because we have $f_i^*(\boldsymbol{x}) \preceq f_i^*(\boldsymbol{x}'')$ by (2.79) and we have $f_i^*(\boldsymbol{x}) \succeq f_i^*(\boldsymbol{x}'')$ by (2.82), and (C) follows from (2.4). Comparing both sides of (2.83), we obtain

  $$f_j(u_1) = \lambda \tag{2.84}$$

  and $f_{\tau_j(u_1)}^*(\mathrm{suff}(\boldsymbol{u})) = \lambda$.

  We now show that (b) does not hold dividing into two cases by whether f_j is injective.

  - If f_j is not injective, then F does not satisfy Definition 2.2.3 (b) by $k = 0$ and Remark 2.2.1.

– If f_j is injective, then by Lemma 2.2.2 (iii)

$$\bar{\mathcal{P}}^0_{F,j} \neq \emptyset \tag{2.85}$$

by (2.13). We see that F does not satisfy Definition 2.2.3 (a) because

$$\begin{aligned}
\mathcal{P}^0_{F,\tau_j(u_1)} \cap \bar{\mathcal{P}}^0_{F,j}(f_j(u_1)) &\overset{\text{(A)}}{=} \mathcal{P}^0_{F,\tau_j(u_1)} \cap \bar{\mathcal{P}}^0_{F,j} &\quad (2.86)\\
&\overset{\text{(B)}}{=} \{\lambda\} \cap \bar{\mathcal{P}}^0_{F,j} &\quad (2.87)\\
&\overset{\text{(C)}}{=} \{\lambda\} \cap \{\lambda\} &\quad (2.88)\\
&= \{\lambda\} &\quad (2.89)\\
&\neq \emptyset, &\quad (2.90)
\end{aligned}$$

where (A) follows from (2.84), (B) follows from (2.16), and (C) follows from (2.85).

- The case $\boldsymbol{x} \not\preceq \boldsymbol{x}''$: By (2.81) and $\boldsymbol{x} \not\preceq \boldsymbol{x}''$, there exist $\boldsymbol{z}, \boldsymbol{z}'' \in \mathcal{S}^+$ such that

$$\begin{aligned}
\boldsymbol{x} &= \boldsymbol{y}\boldsymbol{z}, &\quad (2.91)\\
\boldsymbol{x}'' &= \boldsymbol{y}\boldsymbol{z}'', &\quad (2.92)\\
z_1 &\neq z''_1, &\quad (2.93)
\end{aligned}$$

where $\boldsymbol{y} \coloneqq \boldsymbol{x} \wedge \boldsymbol{x}''$. Defining $\boldsymbol{z}' \coloneqq \boldsymbol{z}\boldsymbol{x}^{-1}\boldsymbol{x}'$, defined by (2.80), we have

$$\boldsymbol{x}' = \boldsymbol{x}\boldsymbol{x}^{-1}\boldsymbol{x}' = \boldsymbol{y}\boldsymbol{z}\boldsymbol{x}^{-1}\boldsymbol{x} = \boldsymbol{y}\boldsymbol{z}', \tag{2.94}$$

$$z_1 = z'_1. \tag{2.95}$$

Then defining $j \coloneqq \tau^*_i(\boldsymbol{y})$, we have

$$\begin{aligned}
&f^*_i(\boldsymbol{y})f_j(z'_1)f^*_{\tau_j(z'_1)}(\mathrm{suff}(\boldsymbol{z}'))\\
&\overset{\text{(A)}}{=} f^*_i(\boldsymbol{y})f^*_j(\boldsymbol{z}') \overset{\text{(B)}}{=} f^*_i(\boldsymbol{y}\boldsymbol{z}') \overset{\text{(C)}}{=} f^*_i(\boldsymbol{x}') \overset{\text{(D)}}{\succeq} f^*_i(\boldsymbol{x})\boldsymbol{c}, \qquad (2.96)
\end{aligned}$$

where (A) follows from (2.4), (B) follows from Lemma 2.1.1 (i), (C) follows from (2.94), and (D) follows from (2.78). Similarly, by (2.79) and (2.92), we have

$$\begin{aligned}
&f^*_i(\boldsymbol{y})f_j(z''_1)f^*_{\tau_j(z''_1)}(\mathrm{suff}(\boldsymbol{z}''))\\
&= f^*_i(\boldsymbol{y})f^*_j(\boldsymbol{z}'') = f^*_i(\boldsymbol{y}\boldsymbol{z}'') = f^*_i(\boldsymbol{x}'') \succeq f^*_i(\boldsymbol{x})\boldsymbol{c}. \qquad (2.97)
\end{aligned}$$

Also, we have

$$f_i^*(\boldsymbol{x})\boldsymbol{c} \overset{\text{(A)}}{=} f_i^*(\boldsymbol{y}\boldsymbol{z})\boldsymbol{c} \overset{\text{(B)}}{=} f_i^*(\boldsymbol{y})f_j^*(\boldsymbol{z})\boldsymbol{c} \overset{\text{(C)}}{=} f_i^*(\boldsymbol{y})f_j(z_1)f^*_{\tau_j(z_1)}(\mathrm{suff}(\boldsymbol{z}))\boldsymbol{c}, \tag{2.98}$$

where (A) follows from (2.91), (B) follows from Lemma 2.1.1 (i), and (C) follows from (2.4).

Thus, we have

$$\begin{aligned} f_j(z_1')f^*_{\tau_j(z_1')}(\mathrm{suff}(\boldsymbol{z}')) &\overset{\text{(A)}}{\succeq} f_j(z_1)f^*_{\tau_j(z_1)}(\mathrm{suff}(\boldsymbol{z}))\boldsymbol{c} &(2.99)\\ &\overset{\text{(B)}}{=} f_j(z_1')f^*_{\tau_j(z_1')}(\mathrm{suff}(\boldsymbol{z}))\boldsymbol{c} &(2.100)\\ &\succeq f_j(z_1')\boldsymbol{c}', &(2.101)\end{aligned}$$

where $\boldsymbol{c}' := [f^*_{\tau_j(z_1')}(\mathrm{suff}(\boldsymbol{z}))\boldsymbol{c}]_k$, (A) follows from (2.96) and (2.98), and (B) follows from (2.95). Similarly, we have

$$\begin{aligned} f_j(z_1'')f^*_{\tau_j(z_1'')}(\mathrm{suff}(\boldsymbol{z}'')) &\overset{\text{(A)}}{\succeq} f_j(z_1)f^*_{\tau_j(z_1)}(\mathrm{suff}(\boldsymbol{z}))\boldsymbol{c} &(2.102)\\ &\overset{\text{(B)}}{=} f_j(z_1')f^*_{\tau_j(z_1')}(\mathrm{suff}(\boldsymbol{z}))\boldsymbol{c} &(2.103)\\ &\succeq f_j(z_1')\boldsymbol{c}', &(2.104)\end{aligned}$$

where (A) follows from (2.97) and (2.98), and (B) follows from (2.95). By (2.101), we have $f^*_{\tau_j(z_1')}(\mathrm{suff}(\boldsymbol{z}')) \succeq \boldsymbol{c}'$, which leads to

$$\boldsymbol{c}' \in \mathcal{P}^k_{F,\tau_j(z_1')} \tag{2.105}$$

by (2.16).

By (2.104), at least one of $f_j(z_1') \preceq f_j(z_1'')$ and $f_j(z_1') \succeq f_j(z_1'')$ holds. We may assume $f_j(z_1') \preceq f_j(z_1'')$ by symmetry. We consider the following two cases separately: the case $f_j(z_1') \prec f_j(z_1'')$ and the case $f_j(z_1') = f_j(z_1'')$.

– the case $f_j(z_1') \prec f_j(z_1'')$: We have

$$f_j^*(\boldsymbol{z}'') \overset{\text{(A)}}{=} f_j(z_1'')f^*_{\tau_j(z_1'')}(\mathrm{suff}(\boldsymbol{z}'')) \overset{\text{(B)}}{\succeq} f_j(z_1')\boldsymbol{c}', \tag{2.106}$$

where (A) follows from (2.4), and (B) follows from (2.104). By (2.106) and $f_j(z_1') \prec f_j(z_1'')$, we obtain

$$\boldsymbol{c}' \in \bar{\mathcal{P}}_{F,j}^{k}(f_j(z_1')) \tag{2.107}$$

by (2.13). By (2.105) and (2.107), the code-tuple F does not satisfy Definition 2.2.3 (a).

– the case $f_j(z_1') = f_j(z_1'')$: We have

$$f_j(z_1') f^*_{\tau_j(z_1'')}(\mathrm{suff}(\boldsymbol{z}'')) \overset{\text{(A)}}{=} f_j(z_1'') f^*_{\tau_j(z_1'')}(\mathrm{suff}(\boldsymbol{z}'')) \overset{\text{(B)}}{\succeq} f_j(z_1')\boldsymbol{c}', \tag{2.108}$$

where (A) follows from $f_j(z_1') = f_j(z_1'')$, and (B) follows from (2.104). This shows $f^*_{\tau_j(z_1'')}(\mathrm{suff}(\boldsymbol{z}'')) \succeq \boldsymbol{c}'$, which leads to

$$\boldsymbol{c}' \in \mathcal{P}_{F,\tau_j(z_1'')}^{k} \tag{2.109}$$

by (2.16). By $f_j(z_1') = f_j(z_1'')$, (2.93), (2.105), and (2.109), the code-tuple F does not satisfy Definition 2.2.3 (b).

□

2.6.2 Proof of Lemma 2.3.3

Proof of Lemma 2.3.3. It suffices to show that $|f_i^*(\boldsymbol{x})| \geq 1$ holds for any $i \in [F]$ and $\boldsymbol{x} \in \mathcal{S}^{|F|}$. We prove by contradiction assuming that there exist $i \in [F]$ and $\boldsymbol{x} = x_1x_2\ldots x_{|F|} \in \mathcal{S}^{|F|}$ such that $f_i^*(\boldsymbol{x}) = \lambda$. Then by pigeonhole principle, we can choose integers p, q such that $0 \leq p < q \leq |F|$ and

$$\tau_i^*(x_1x_2\ldots x_p) = \tau_i^*(x_1x_2\ldots x_q) =: j. \tag{2.110}$$

We have

$$\tau_j^*(x_{p+1}x_{p+2}\ldots x_q) \overset{\text{(A)}}{=} \tau^*_{\tau_i^*(x_1x_2\ldots x_p)}(x_{p+1}x_{p+2}\ldots x_q) \overset{\text{(B)}}{=} \tau_i^*(x_1x_2\ldots x_q) \overset{\text{(C)}}{=} j, \tag{2.111}$$

where (A) follows from (2.110), (B) follows from Lemma 2.1.1 (ii), and (C) follows from (2.110). Thus, we obtain

$$\mathcal{P}^k_{F,\tau_j(x_{p+1})} \overset{\text{(A)}}{\supseteq} \mathcal{P}^k_{F,\tau_j^*(x_{p+1}x_{p+2})} \overset{\text{(A)}}{\supseteq} \cdots \overset{\text{(A)}}{\supseteq} \mathcal{P}^k_{F,\tau_j^*(x_{p+1}x_{p+2}\ldots x_q)} \overset{\text{(B)}}{=} \mathcal{P}^k_{F,j}, \tag{2.112}$$

where (A)s follow from Lemma 2.2.1 (i) and (ii) and $f_i^*(\boldsymbol{x}) = \lambda$, and (B) follows from (2.111).

We consider the following two cases separately: the case $\bar{\mathcal{P}}^k_{F,j} \neq \emptyset$ and the case $\bar{\mathcal{P}}^k_{F,j} = \emptyset$.

- The case $\bar{\mathcal{P}}^k_{F,j} \neq \emptyset$: We have

$$\mathcal{P}^k_{F,\tau_j(x_{p+1})} \cap \bar{\mathcal{P}}^k_{F,j} \overset{\text{(A)}}{\supseteq} \mathcal{P}^k_{F,j} \cap \bar{\mathcal{P}}^k_{F,j} \overset{\text{(B)}}{\supseteq} \bar{\mathcal{P}}^k_{F,j} \cap \bar{\mathcal{P}}^k_{F,j} = \bar{\mathcal{P}}^k_{F,j} \overset{\text{(C)}}{\neq} \emptyset, \quad (2.113)$$

  where (A) follows from (2.112), (B) follows from Lemma 2.2.1 (i), and (C) follows from the assumption. Therefore, F does not satisfy Definition 2.2.3 (a), which conflicts with $F \in \mathscr{F}_{k\text{-dec}}$.

- The case $\bar{\mathcal{P}}^k_{F,j} = \emptyset$: By Corollary 2.3.1 (ii) (a), we have $\bar{\mathcal{P}}^0_{F,j} = \emptyset$. Hence, by Lemma 2.2.2 (iii), we have $|\mathcal{S}| \geq 2$ so that there exists $s \in \mathcal{S}$ such that $s \neq x_{p+1}$ and $f_j(s) = \lambda = f_j(x_{p+1})$. We have

$$\mathcal{P}^k_{F,\tau_j(x_{p+1})} \cap \mathcal{P}^k_{F,\tau_j(s)} \overset{\text{(A)}}{\supseteq} \mathcal{P}^k_{F,j} \cap \mathcal{P}^k_{F,\tau_j(s)} \quad (2.114)$$
$$\overset{\text{(B)}}{\supseteq} \mathcal{P}^k_{F,\tau_j(s)} \cap \mathcal{P}^k_{F,\tau_j(s)} \quad (2.115)$$
$$= \mathcal{P}^k_{F,\tau_j(s)} \quad (2.116)$$
$$\overset{\text{(C)}}{\neq} \emptyset, \quad (2.117)$$

  where (A) follows from (2.112), (B) follows from Lemma 2.2.1 (i) and (ii), and (C) follows from $F \in \mathscr{F}_{\text{ext}}$ and Corollary 2.3.2 (i) (a). Therefore, F does not satisfy Definition 2.2.3 (b), which conflicts with $F \in \mathscr{F}_{k\text{-dec}}$.

□

2.6.3 Proof of Lemma 2.4.1

In preparation for the proof, we introduce the following Definition 2.6.1 and Lemma 2.6.1.

Definition 2.6.1. *Let $F(f, \tau) \in \mathscr{F}$. A set $\mathcal{I} \subseteq [F]$ is said to be* closed *if for any $i \in \mathcal{I}$ and $s \in \mathcal{S}$, it holds that $\tau_i(s) \in \mathcal{I}$.*

Lemma 2.6.1. *For any $F \in \mathscr{F}$ and $\boldsymbol{x} = (x_0, x_1, \ldots, x_{|F|-1}) \in \mathbb{R}^{|F|}$, if*

$$\boldsymbol{x} Q(F) = \boldsymbol{x}, \tag{2.118}$$

then both of $\mathcal{I}_+ \coloneqq \{i \in [F] : x_i > 0\}$ and $\mathcal{I}_- \coloneqq \{i \in [F] : x_i < 0\}$ are closed.

Proof of Lemma 2.6.1. By symmetry, it suffices to prove only that $\mathcal{I}_+$ is closed. We have

$$\begin{aligned}
&\sum_{i \in \mathcal{I}_+} \sum_{j \in \mathcal{I}_+} x_j Q_{j,i}(F) + \sum_{i \in \mathcal{I}_+} \sum_{j \in [F] \setminus \mathcal{I}_+} x_j Q_{j,i}(F) \\
&= \sum_{i \in \mathcal{I}_+} \sum_{j \in [F]} x_j Q_{j,i}(F) && (2.119) \\
&\overset{\text{(A)}}{=} \sum_{i \in \mathcal{I}_+} x_i && (2.120) \\
&\overset{\text{(B)}}{=} \sum_{i \in \mathcal{I}_+} x_i \sum_{j \in [F]} Q_{i,j}(F) && (2.121) \\
&= \sum_{i \in \mathcal{I}_+} \sum_{j \in [F]} x_i Q_{i,j}(F) && (2.122) \\
&= \sum_{i \in \mathcal{I}_+} \sum_{j \in \mathcal{I}_+} x_i Q_{i,j}(F) + \sum_{i \in \mathcal{I}_+} \sum_{j \in [F] \setminus \mathcal{I}_+} x_i Q_{i,j}(F) && (2.123) \\
&\overset{\text{(C)}}{=} \sum_{i \in \mathcal{I}_+} \sum_{j \in \mathcal{I}_+} x_j Q_{j,i}(F) + \sum_{i \in \mathcal{I}_+} \sum_{j \in [F] \setminus \mathcal{I}_+} x_i Q_{i,j}(F), && (2.124)
\end{aligned}$$

where (A) follows from (2.118), (B) follows from $\sum_{j \in [F]} Q_{i,j}(F) = 1$ for any $i \in [F]$, and (C) is obtained by exchanging the roles of i and j in the first term. Therefore, we have

$$0 \overset{\text{(A)}}{\geq} \sum_{i \in \mathcal{I}_+} \sum_{j \in [F] \setminus \mathcal{I}_+} x_j Q_{j,i}(F) \overset{\text{(B)}}{=} \sum_{i \in \mathcal{I}_+} \sum_{j \in [F] \setminus \mathcal{I}_+} x_i Q_{i,j}(F) \overset{\text{(C)}}{\geq} 0, \tag{2.125}$$

where (A) follows since $x_j \leq 0$ for any $j \in [F] \setminus \mathcal{I}_+$, (B) is obtained by eliminating the first terms from the leftmost and rightmost sides of (2.124), and (C) follows since $x_i > 0$ for any $i \in \mathcal{I}_+$. This shows

$$\sum_{i \in \mathcal{I}_+} \sum_{j \in [F] \setminus \mathcal{I}_+} x_i Q_{i,j}(F) = 0. \tag{2.126}$$

Since $x_i > 0$ holds for any $i \in \mathcal{I}_+$, it must hold that $Q_{i,j}(F) = 0$ for any $i \in \mathcal{I}_+$ and $j \in [F] \setminus \mathcal{I}_+$. This implies that for any $i \in \mathcal{I}_+$ and $s \in \mathcal{S}$, we have $\tau_i(s) \in \mathcal{I}_+$; that is, $\mathcal{I}_+$ is closed as desired. □

Proof of Lemma 2.4.1. Equation (2.56) can be rewritten as

$$\boldsymbol{\pi} A = \mathbf{0}, \tag{2.127}$$

where $A = (A_{i,j}) \coloneqq Q(F) - E$ and E is the identity matrix. We have $\det A = 0$ because the sum of each row of A equals 0: for any $i \in [F]$, we have

$$\sum_{j\in[F]} A_{i,j} = \sum_{j\in[F]} (Q_{i,j}(F) - \delta_{ij}) \tag{2.128}$$

$$= \sum_{j\in[F]} Q_{i,j}(F) - \sum_{j\in[F]} \delta_{ij} \tag{2.129}$$

$$= \sum_{j\in[F]} Q_{i,j}(F) - 1 \tag{2.130}$$

$$= \sum_{j\in[F]} \sum_{s\in\mathcal{S},\tau_i(s)=j} \mu(s) - 1 \tag{2.131}$$

$$= \sum_{s\in\mathcal{S}} \mu(s) - 1 \tag{2.132}$$

$$= 0, \tag{2.133}$$

where δ_{ij} denotes Kronecker delta. Thus, the dimension of the null space of A is greater than or equal to 1. In particular, Equation (2.127), which is equivalent to (2.56), has a non-trivial solution $\boldsymbol{\pi} \neq \mathbf{0}$. We choose such $\boldsymbol{\pi} = (\pi_0, \pi_1, \ldots, \pi_{|F|-1}) \neq \mathbf{0}$. Then both of $\mathcal{I}_+ \coloneqq \{i \in [F] : \pi_i > 0\}$ and $\mathcal{I}_- \coloneqq \{i \in [F] : \pi_i < 0\}$ are closed by Lemma 2.6.1. Hence, we have

$$^\forall i \in \mathcal{I}_+; {}^\forall j \in [F] \setminus \mathcal{I}_+; Q_{i,j}(F) = 0, \tag{2.134}$$

$$^\forall i \in \mathcal{I}_-; {}^\forall j \in [F] \setminus \mathcal{I}_-; Q_{i,j}(F) = 0. \tag{2.135}$$

Since $\boldsymbol{\pi} \neq \mathbf{0}$, we have $\sum_{i\in[F]} |\pi_i| > 0$ and thus we can define $\boldsymbol{\pi}' = (\pi'_0, \pi'_1, \ldots, \pi'_{|F|-1}) \in \mathbb{R}^{|F|}$ as

$$\pi'_i = \frac{|\pi_i|}{\sum_{i\in[F]} |\pi_i|} \tag{2.136}$$

for $i \in [F]$. This vector $\boldsymbol{\pi}'$ is a desired stationary distribution of F. In fact, by the definition, $\boldsymbol{\pi}'$ clearly satisfies (2.57) and $\pi'_i \geq 0$ for any $i \in [F]$. Also, we can see that $\boldsymbol{\pi}'$ satisfies (2.56) because for any $j \in [F]$, we have

$$\Big(\sum_{i\in[F]} |\pi_i|\Big)\Big(\sum_{i\in[F]} \pi'_i Q_{i,j}(F)\Big) \tag{2.137}$$

$$\overset{\text{(A)}}{=} \sum_{i\in[F]} |\pi_i| Q_{i,j}(F) \tag{2.138}$$

$$= \sum_{i\in\mathcal{I}_+} \pi_i Q_{i,j}(F) - \sum_{i\in\mathcal{I}_-} \pi_i Q_{i,j}(F) \tag{2.139}$$

$$\overset{\text{(B)}}{=} \begin{cases} \sum_{i\in\mathcal{I}_+} \pi_i Q_{i,j}(F) & \text{if } j \in \mathcal{I}_+, \\ -\sum_{i\in\mathcal{I}_-} \pi_i Q_{i,j}(F) & \text{if } j \in \mathcal{I}_-, \\ 0 & \text{otherwise,} \end{cases} \tag{2.140}$$

$$\overset{\text{(C)}}{=} \begin{cases} \sum_{i\in\mathcal{I}_+} \pi_i Q_{i,j}(F) + \sum_{i\in\mathcal{I}_-} \pi_i Q_{i,j}(F) & \text{if } j \in \mathcal{I}_+, \\ -\sum_{i\in\mathcal{I}_+} \pi_i Q_{i,j}(F) - \sum_{i\in\mathcal{I}_-} \pi_i Q_{i,j}(F) & \text{if } j \in \mathcal{I}_-, \\ 0 & \text{otherwise,} \end{cases} \tag{2.141}$$

$$= \begin{cases} \sum_{i\in[F]} \pi_i Q_{i,j}(F) & \text{if } j \in \mathcal{I}_+, \\ -\sum_{i\in[F]} \pi_i Q_{i,j}(F) & \text{if } j \in \mathcal{I}_-, \\ 0 & \text{otherwise,} \end{cases} \tag{2.142}$$

$$\overset{\text{(D)}}{=} \begin{cases} \pi_j & \text{if } j \in \mathcal{I}_+, \\ -\pi_j & \text{if } j \in \mathcal{I}_-, \\ 0 & \text{otherwise,} \end{cases} \tag{2.143}$$

$$= |\pi_j| \tag{2.144}$$

$$\overset{\text{(E)}}{=} \Big(\sum_{i\in[F]} |\pi_i|\Big)\pi'_j, \tag{2.145}$$

where (A) follows from (2.136), (B) follows from (2.134) and (2.135), (C) follows from (2.134) and (2.135), (D) follows since $\boldsymbol{\pi}$ is a stationary distribution of F, and (E) follows from (2.136). □

2.6.4 Proof of Lemma 2.5.1

Proof of Lemma 2.5.1. (Proof of (i)): We first show that $f_i'^*(\boldsymbol{x}) = f_{\varphi(i)}^*(\boldsymbol{x})$ for any $i \in [F']$ and $\boldsymbol{x} \in \mathcal{S}^*$ by induction for $|\boldsymbol{x}|$. For the base case $|\boldsymbol{x}| = 0$, we have $f_i'^*(\lambda) = \lambda = f_{\varphi(i)}^*(\lambda)$ by (2.4). We consider the induction step for $|\boldsymbol{x}| \geq 1$. We have

$$f_i'^*(\boldsymbol{x}) \overset{\text{(A)}}{=} f_i'(x_1) f_{\tau_i'(x_1)}'^*(\mathrm{suff}(\boldsymbol{x})) \tag{2.146}$$

$$\overset{\text{(B)}}{=} f_{\varphi(i)}(x_1) f_{\tau_i'(x_1)}'^*(\mathrm{suff}(\boldsymbol{x})) \tag{2.147}$$

$$\overset{\text{(C)}}{=} f_{\varphi(i)}(x_1) f_{\varphi(\tau_i'(x_1))}^*(\mathrm{suff}(\boldsymbol{x})) \tag{2.148}$$

$$\overset{\text{(D)}}{=} f_{\varphi(i)}(x_1) f_{\tau_{\varphi(i)}(x_1)}^*(\mathrm{suff}(\boldsymbol{x})) \tag{2.149}$$

$$\overset{\text{(E)}}{=} f_{\varphi(i)}^*(\boldsymbol{x}) \tag{2.150}$$

as desired, where (A) follows from (2.4), (B) follows from (2.64), (C) follows from the induction hypothesis, (D) follows from (2.65), and (E) follows from (2.4).

Next, we show that $\varphi(\tau_i'^*(\boldsymbol{x})) = \tau_{\varphi(i)}^*(\boldsymbol{x})$ for any $i \in [F']$ and $\boldsymbol{x} \in \mathcal{S}^*$ by induction for $|\boldsymbol{x}|$. For the base case $|\boldsymbol{x}| = 0$, we have $\varphi(\tau_i'^*(\lambda)) = \varphi(i) = \tau_{\varphi(i)}^*(\lambda)$ by (2.5). We consider the induction step for $|\boldsymbol{x}| \geq 1$. We have

$$\varphi(\tau_i'^*(\boldsymbol{x})) \overset{\text{(A)}}{=} \varphi(\tau_{\tau_i'(x_1)}'^*(\mathrm{suff}(\boldsymbol{x}))) \overset{\text{(B)}}{=} \tau_{\varphi(\tau_i'(x_1))}^*(\mathrm{suff}(\boldsymbol{x})) \tag{2.151}$$

$$\overset{\text{(C)}}{=} \tau_{\tau_{\varphi(i)}(x_1)}^*(\mathrm{suff}(\boldsymbol{x})) \overset{\text{(D)}}{=} \tau_{\varphi(i)}^*(\boldsymbol{x}) \tag{2.152}$$

as desired, where (A) follows from (2.5), (B) follows from the induction hypothesis, (C) follows from (2.65), and (D) follows from (2.5).

(Proof of (ii)): For any $\boldsymbol{c} \in \mathcal{C}^*$, we have

$$\boldsymbol{c} \in \mathcal{P}_{F',i}^*(\boldsymbol{b}) \overset{\text{(A)}}{\iff} {}^{\exists}\boldsymbol{x} \in \mathcal{S}^+; (f_i'^*(\boldsymbol{x}) \succeq \boldsymbol{bc}, f_i'(x_1) \succeq \boldsymbol{b}) \tag{2.153}$$

$$\overset{\text{(B)}}{\iff} {}^{\exists}\boldsymbol{x} \in \mathcal{S}^+; (f_{\varphi(i)}^*(\boldsymbol{x}) \succeq \boldsymbol{bc}, f_{\varphi(i)}(x_1) \succeq \boldsymbol{b}) \tag{2.154}$$

$$\overset{\text{(C)}}{\iff} \boldsymbol{c} \in \mathcal{P}_{F,\varphi(i)}^*(\boldsymbol{b}), \tag{2.155}$$

where (A) follows from (2.12), (B) follows from (i) of this lemma, and (C) follows from (2.12). This shows that $\mathcal{P}_{F',i}^*(\boldsymbol{b}) = \mathcal{P}_{F,\varphi(i)}^*(\boldsymbol{b})$. We can prove $\bar{\mathcal{P}}_{F',i}^*(\boldsymbol{b}) = \bar{\mathcal{P}}_{F,\varphi(i)}^*(\boldsymbol{b})$ by the same way using (2.13).

(Proof of (iii)): For any $i' \in [F']$ and $j \in [F]$, we have

$$\sum_{j' \in \mathcal{A}_j} Q_{i',j'}(F') \overset{\text{(A)}}{=} \sum_{j' \in \mathcal{A}_j} \sum_{\substack{s \in \mathcal{S} \\ \tau'_{i'}(s) = j'}} \mu(s) \tag{2.156}$$

$$= \sum_{\substack{s \in \mathcal{S} \\ \tau'_{i'}(s) \in \mathcal{A}_j}} \mu(s) \tag{2.157}$$

$$\overset{\text{(B)}}{=} \sum_{\substack{s \in \mathcal{S} \\ \varphi(\tau'_{i'}(s)) = j}} \mu(s) \tag{2.158}$$

$$\overset{\text{(C)}}{=} \sum_{\substack{s \in \mathcal{S} \\ \tau_{\varphi(i')}(s) = j}} \mu(s) \tag{2.159}$$

$$\overset{\text{(D)}}{=} Q_{\varphi(i'),j}(F) \tag{2.160}$$

$$= Q_{i,j}(F), \tag{2.161}$$

where $i \coloneqq \varphi(i')$ and (A) follows from (2.54), (B) follows from (2.67), (C) follows from (2.65), and (D) follows from (2.54). Thus, for any $j \in [F]$, we have

$$\pi_j = \sum_{j' \in \mathcal{A}_j} \pi'_{j'} \tag{2.162}$$

$$\overset{\text{(A)}}{=} \sum_{j' \in \mathcal{A}_j} \sum_{i' \in [F']} \pi'_{i'} Q_{i',j'}(F') \tag{2.163}$$

$$= \sum_{j' \in \mathcal{A}_j} \sum_{i \in [F]} \sum_{i' \in \mathcal{A}_i} \pi'_{i'} Q_{i',j'}(F') \tag{2.164}$$

$$= \sum_{i \in [F]} \sum_{i' \in \mathcal{A}_i} \pi'_{i'} \sum_{j' \in \mathcal{A}_j} Q_{i',j'}(F') \tag{2.165}$$

$$\overset{\text{(B)}}{=} \sum_{i \in [F]} \sum_{i' \in \mathcal{A}_i} \pi'_{i'} Q_{i,j}(F) \tag{2.166}$$

$$= \sum_{i \in [F]} Q_{i,j}(F) \sum_{i' \in \mathcal{A}_i} \pi'_{i'} \tag{2.167}$$

$$= \sum_{i \in [F]} Q_{i,j}(F) \pi_i, \tag{2.168}$$

where (A) follows since $\boldsymbol{\pi}'$ satisfies (2.56), and (B) follows from (2.161) and $i' \in \mathcal{A}_i$. Also, we have

$$\sum_{i\in[F]} \pi_i = \sum_{i\in[F]} \sum_{i'\in\mathcal{A}_i} \pi'_{i'} = \sum_{i'\in[F']} \pi'_{i'} \overset{\text{(A)}}{=} 1, \tag{2.169}$$

where (A) follows since $\boldsymbol{\pi}'$ satisfies (2.57). By (2.168) and (2.169), $\boldsymbol{\pi}$ is a stationary distribution of F.

(Proof of (iv)): We have

$$F \in \mathscr{F}_{\text{ext}} \iff {}^{\forall} i \in [F]; \mathcal{P}^1_{F,i} \neq \emptyset \tag{2.170}$$
$$\implies {}^{\forall} i' \in [F']; \mathcal{P}^1_{F,\varphi(i')} \neq \emptyset \tag{2.171}$$
$$\overset{\text{(A)}}{\iff} {}^{\forall} i' \in [F']; \mathcal{P}^1_{F',i'} \neq \emptyset \tag{2.172}$$
$$\iff F' \in \mathscr{F}_{\text{ext}}, \tag{2.173}$$

where (A) follows from (ii) of this lemma.

(Proof of (v)): By $F, F' \in \mathscr{F}_{\text{reg}}$, the code-tuples F and F' have the unique stationary distributions $\boldsymbol{\pi}(F)$ and $\boldsymbol{\pi}(F')$, respectively. By (iii) of this lemma, we have

$${}^{\forall} j \in [F]; \pi_j(F) = \sum_{j'\in\mathcal{A}_j} \pi_{j'}(F'), \tag{2.174}$$

where

$$\mathcal{A}_i \coloneqq \{i' \in [F'] : \varphi(i') = i\} \tag{2.175}$$

for $i \in [F]$. Therefore, we have

$$L(F') = \sum_{i' \in [F']} \pi_{i'}(F') L_{i'}(F') \tag{2.176}$$

$$= \sum_{i \in [F]} \sum_{i' \in \mathcal{A}_i} \pi_{i'}(F') L_{i'}(F') \tag{2.177}$$

$$\overset{\text{(A)}}{=} \sum_{i \in [F]} \sum_{i' \in \mathcal{A}_i} \pi_{i'}(F') L_{\varphi(i')}(F) \tag{2.178}$$

$$\overset{\text{(B)}}{=} \sum_{i \in [F]} \sum_{i' \in \mathcal{A}_i} \pi_{i'}(F') L_i(F) \tag{2.179}$$

$$= \sum_{i \in [F]} L_i(F) \sum_{i' \in \mathcal{A}_i} \pi_{i'}(F') \tag{2.180}$$

$$\overset{\text{(C)}}{=} \sum_{i \in [F]} \pi_i(F) L_i(F) \tag{2.181}$$

$$= L(F) \tag{2.182}$$

as desired, where (A) follows from (2.64) (cf. Remark 2.4.1), (B) follows from (2.175) and $i' \in \mathcal{A}_i$, and (C) follows from (2.174).

(Proof of (vi)): For any $i \in [F']$ and $s \in \mathcal{S}$, we have

$$\mathcal{P}^k_{F',\tau'_i(s)} \cap \bar{\mathcal{P}}^k_{F',i}(f'_i(s)) \overset{\text{(A)}}{=} \mathcal{P}^k_{F,\varphi(\tau'_i(s))} \cap \bar{\mathcal{P}}^k_{F,\varphi(i)}(f'_i(s)) \tag{2.183}$$

$$\overset{\text{(B)}}{=} \mathcal{P}^k_{F,\tau_{\varphi(i)}(s)} \cap \bar{\mathcal{P}}^k_{F,\varphi(i)}(f_{\varphi(i)}(s)) \tag{2.184}$$

$$\overset{\text{(C)}}{=} \emptyset, \tag{2.185}$$

where (A) follows from (ii) of this lemma, (B) follows from (2.64) and (2.65), and (C) follows from $F \in \mathscr{F}_{k\text{-dec}}$. Namely, F' satisfies Definition 2.2.3 (a).

Choose $i \in [F']$ and $s, s' \in \mathcal{S}$ such that $s \neq s'$ and $f'_i(s) = f'_i(s')$ arbitrarily. Then by (2.64), we have

$$f_{\varphi(i)}(s) = f'_i(s) = f'_i(s') = f_{\varphi(i)}(s'). \tag{2.186}$$

Thus, we obtain

$$\mathcal{P}^k_{F',\tau'_i(s)} \cap \mathcal{P}^k_{F',\tau'_i(s')} \overset{\text{(A)}}{=} \mathcal{P}^k_{F,\varphi(\tau'_i(s))} \cap \mathcal{P}^k_{F,\varphi(\tau'_i(s'))} \overset{\text{(B)}}{=} \mathcal{P}^k_{F,\tau_{\varphi(i)}(s)} \cap \mathcal{P}^k_{F,\tau_{\varphi(i)}(s')} \overset{\text{(C)}}{=} \emptyset, \tag{2.187}$$

where (A) follows from (ii) of this lemma, (B) follows from (2.65), (C) follows from (2.186) and $F \in \mathscr{F}_{k\text{-dec}}$. Namely, F' satisfies Definition 2.2.3 (b). □

2.6.5 Proof of Lemma 2.5.2

To prove Lemma 2.5.2, we first prove the following Lemmas 2.6.2–2.6.4. Lemmas 2.6.2 and 2.6.3 relate to closed sets defined in Subsection 2.6.3.

Lemma 2.6.2. *For any $F \in \mathscr{F}$, the following statements (i) and (ii) hold.*

(i) $\mathcal{R}_F$ is closed.

(ii) For any non-empty closed set $\mathcal{I} \subseteq [F]$, we have $\mathcal{R}_F \subseteq \mathcal{I}$.

Proof of Lemma 2.6.2. (Proof of (i)): Choose $i \in \mathcal{R}_F$ and $s \in \mathcal{S}$ arbitrarily. For any $j \in [F]$, there exists $\boldsymbol{x} \in \mathcal{S}^*$ such that $\tau_j^*(\boldsymbol{x}) = i$, which leads to

$$\tau_j^*(\boldsymbol{x}s) \stackrel{\text{(A)}}{=} \tau_{\tau_j^*(\boldsymbol{x})}(s) = \tau_i(s), \tag{2.188}$$

where (A) follows from Lemma 2.1.1 (ii). This shows $\tau_i(s) \in \mathcal{R}_F$.

(Proof of (ii)): Choose $i \in \mathcal{R}_F$ arbitrarily. We prove $i \in \mathcal{I}$ by contradiction assuming the contrary $i \notin \mathcal{I}$. Since $\mathcal{I} \neq \emptyset$, we can choose $j \in \mathcal{I}$. By $i \in \mathcal{R}_F$, there exists $\boldsymbol{x} = x_1x_2\ldots x_n \in \mathcal{S}^*$ such that $\tau_j^*(\boldsymbol{x}) = i$. We define $i_l \coloneqq \tau_j^*(x_1x_2\ldots x_l)$ for $l = 0, 1, 2, \ldots, n$. Since $i_0 = \tau_j^*(\lambda) = j \in \mathcal{I}$ and $i_n = \tau_j^*(\boldsymbol{x}) = i \notin \mathcal{I}$, there exists an integer $0 \leq l < n$ such that $i_l \in \mathcal{I}$ and $i_{l+1} = \tau_{i_l}(x_{l+1}) \notin \mathcal{I}$. This conflicts with that $\mathcal{I}$ is closed. □

Lemma 2.6.3. *For any $F \in \mathscr{F}$ and non-empty closed set $\mathcal{I} \subseteq [F]$, the following statements (i) and (ii) hold.*

(i) There exist $F' \in \mathscr{F}^{(|\mathcal{I}|)}$ and an injective homomorphism $\varphi : [F'] \to [F]$ from F' to F such that $\mathcal{I} = \varphi([F']) \coloneqq \{\varphi(i) : i \in [F']\}$.

(ii) There exists a stationary distribution $\boldsymbol{\pi} = (\pi_0, \pi_1, \ldots, \pi_{|F|-1})$ of F such that $\pi_i = 0$ for any $i \in [F] \setminus \mathcal{I}$.

Proof of Lemma 2.6.3. (Proof of (i)): Suppose $\mathcal{I} = \{i_0, i_1, \ldots, i_{m-1}\}$, where $i_0 < i_1 < \cdots < i_{m-1}$ and $m = |\mathcal{I}|$. We define a mapping $\varphi : [m] \to [F]$ as $\varphi(j) = i_j$ for $j \in [m]$. Since φ is injective and $\varphi([m]) = \mathcal{I}$, we can consider the inverse mapping $\varphi^{-1} : \mathcal{I} \to [m]$, which maps $\varphi(i)$ to i for any $i \in [m]$. Also, we define $F'(f', \tau') \in \mathscr{F}^{(m)}$ as

$$f'_i(s) = f_{\varphi(i)}(s), \tag{2.189}$$

$$\tau'_i(s) = \varphi^{-1}(\tau_{\varphi(i)}(s)) \tag{2.190}$$

for $i \in [F']$ and $s \in \mathcal{S}$. Since $\mathcal{I}$ is closed, we have $\tau_{\varphi(i)}(s) \in \mathcal{I}$ and thus $\tau'_i(s) = \varphi^{-1}(\tau_{\varphi(i)}(s)) \in [m] = [F']$; that is, F' is indeed well-defined. We can see that φ is a homomorphism from F' to F directly from (2.189) and (2.190).

(Proof of (ii)): By (i) of this lemma, there exist $F' \in \mathscr{F}$ and an injective homomorphism $\varphi : [F'] \to [F]$ from F' to F such that

$$\varphi([F']) = \mathcal{I}. \tag{2.191}$$

By Lemma 2.4.1, we can choose a stationary distribution $\boldsymbol{\pi}'$ of F'. By Lemma 2.5.1 (iii), the vector $\boldsymbol{\pi} \in \mathbb{R}^{|F|}$ defined as (2.66) is a stationary distribution of F. This vector $\boldsymbol{\pi}$ is a desired stationary distribution because $\mathcal{A}_i = \{i' \in [F'] : \varphi(i') = i\} = \emptyset$ holds for any $i \in [F] \setminus \mathcal{I}$ by (2.191). □

Lemma 2.6.4. *For any $F \in \mathscr{F}$, If $\mathcal{R}_F = \emptyset$, then there exist $p, q \in [F]$ such that $\mathcal{I}_p \cap \mathcal{I}_q = \emptyset$, where $\mathcal{I}_i := \{\tau_i^*(\boldsymbol{x}) : \boldsymbol{x} \in \mathcal{S}^*\}$ for $i \in [F]$.*

Proof of Lemma 2.6.4. We first show that for any $i, j \in [F]$, we have

$$j \in \mathcal{I}_i \implies \mathcal{I}_j \subseteq \mathcal{I}_i. \tag{2.192}$$

Assume $j \in \mathcal{I}_i$ and choose $p \in \mathcal{I}_j$ arbitrarily. Then there exists $\boldsymbol{x} \in \mathcal{S}^*$ such that $\tau_j^*(\boldsymbol{x}) = p$. Also, by $j \in \mathcal{I}_i$, there exists $\boldsymbol{y} \in \mathcal{S}^*$ such that $\tau_i^*(\boldsymbol{y}) = j$. Therefore, we have

$$\tau_i^*(\boldsymbol{yx}) \overset{\text{(A)}}{=} \tau^*_{\tau_i^*(\boldsymbol{y})}(\boldsymbol{x}) = \tau_j^*(\boldsymbol{x}) = p, \tag{2.193}$$

where (A) follows from Lemma 2.1.1 (ii). This leads to $p \in \mathcal{I}_i$ and thus we obtain (2.192).

Now, we prove Lemma 2.6.4 by proving its contraposition. Namely, we show $\mathcal{R}_F \neq \emptyset$ assuming that

$$^\forall i, j \in [F]; \mathcal{I}_i \cap \mathcal{I}_j \neq \emptyset. \tag{2.194}$$

We can see that

$$\mathcal{R}_F = \bigcap_{i \in [F]} \mathcal{I}_i \tag{2.195}$$

because for any $j \in [F]$, it holds that

$$\begin{aligned} j \in \bigcap_{i \in [F]} \mathcal{I}_i &\iff {}^\forall i \in [F]; j \in \mathcal{I}_i && (2.196) \\ &\iff {}^\forall i \in [F]; {}^\exists \boldsymbol{x} \in \mathcal{S}^*; \tau_i^*(\boldsymbol{x}) = j && (2.197) \\ &\iff j \in \mathcal{R}_F. && (2.198) \end{aligned}$$

Thus, to show $\mathcal{R}_F \neq \emptyset$, it suffices to show that

$$\bigcap_{i \in [r]} \mathcal{I}_i \neq \emptyset \tag{2.199}$$

for any $r = 1, 2, \ldots, |F|$ since the case $r = |F|$ gives the desired result.

We prove (2.199) by induction for r. The base case $r = 1$ is trivial since $\mathcal{I}_0 \ni 0$. We consider the induction step for $r \geq 2$. By the induction hypothesis, we have $\bigcap_{i \in [r-1]} \mathcal{I}_i \neq \emptyset$. Therefore, we can choose $j \in [F]$ such that $j \in \mathcal{I}_i$ for any $i \in [r-1]$. By (2.192), we have $\mathcal{I}_j \subseteq \mathcal{I}_i$ for any $i \in [r-1]$ and thus

$$\mathcal{I}_j \subseteq \bigcap_{i \in [r-1]} \mathcal{I}_i. \tag{2.200}$$

Hence, we obtain

$$\bigcap_{i \in [r]} \mathcal{I}_i = \Big(\bigcap_{i \in [r-1]} \mathcal{I}_i \Big) \cap \mathcal{I}_{r-1} \overset{\text{(A)}}{\supseteq} \mathcal{I}_j \cap \mathcal{I}_{r-1} \overset{\text{(B)}}{\neq} \emptyset \tag{2.201}$$

as desired, where (A) follows from (2.200), and (B) follows from (2.194). □

Proof of Lemma 2.5.2. (Proof of (i)): (Necessity) We assume $\mathcal{R}_F = \emptyset$ and show that F has two distinct stationary distributions. By Lemma 2.6.4, we can choose $p, q \in [F]$ such that

$$\mathcal{I}_p \cap \mathcal{I}_q = \emptyset. \tag{2.202}$$

We can see that $\mathcal{I}_p$ is not empty since $\mathcal{I}_p \ni p$ and also see that $\mathcal{I}_p$ is closed because for any $i \in \mathcal{I}_p$, we have

$$\{\tau_i(s) : s \in \mathcal{S}\} \subseteq \{\tau_i^*(\boldsymbol{x}) : \boldsymbol{x} \in \mathcal{S}^*\} = \mathcal{I}_i \overset{\text{(A)}}{\subseteq} \mathcal{I}_p, \tag{2.203}$$

where (A) follows from (2.192). By the same argument, also $\mathcal{I}_q$ is a non-empty closed set. Therefore, by Lemma 2.6.3 (ii), there exist stationary distributions $\boldsymbol{\pi} = (\pi_0, \pi_1, \ldots, \pi_{|F|-1})$ and $\boldsymbol{\pi}' = (\pi'_0, \pi'_1, \ldots, \pi'_{|F|-1})$ of F such that

$$^\forall i \in [F] \setminus \mathcal{I}_p ; \pi_i = 0 \tag{2.204}$$

and

$$^\forall i \in [F] \setminus \mathcal{I}_q ; \pi'_i = 0. \tag{2.205}$$

Since $\boldsymbol{\pi}$ satisfies (2.57), we have $\pi_j > 0$ for some $j \in [F]$. By (2.204) and (2.202), it must hold that $j \in \mathcal{I}_p \subseteq [F] \setminus \mathcal{I}_q$. Hence, we obtain $\pi'_j = 0 < \pi_j$ by (2.205). This shows $\boldsymbol{\pi} \neq \boldsymbol{\pi}'$. Therefore, we conclude that F has two distinct stationary distributions as desired.

(Sufficiency) We prove $\mathcal{R}_F = \emptyset$ assuming that there exist two distinct stationary distributions $\boldsymbol{\pi} = (\pi_0, \pi_1, \ldots, \pi_{|F|-1})$ and $\boldsymbol{\pi}' = (\pi'_0, \pi'_1, \ldots, \pi'_{|F|-1})$ of F. Then $\boldsymbol{x} = (x_0, x_1, \ldots, x_{|F|-1}) \coloneqq \boldsymbol{\pi} - \boldsymbol{\pi}' \neq \mathbf{0}$ satisfies

$$\boldsymbol{x}Q(F) = \boldsymbol{\pi}Q(F) - \boldsymbol{\pi}'Q(F) \stackrel{\text{(A)}}{=} \boldsymbol{\pi} - \boldsymbol{\pi}' = \boldsymbol{x}, \tag{2.206}$$

$$\sum_{i \in [F]} x_i = \sum_{i \in [F]} \pi_i - \sum_{i \in [F]} \pi'_i \stackrel{\text{(B)}}{=} 1 - 1 = 0, \tag{2.207}$$

where (A) follows from (2.56), and (B) follows from (2.57). Thus, by $\boldsymbol{x} \neq \mathbf{0}$ and (2.207), both of $\mathcal{I}_+ \coloneqq \{i \in [F] : x_i > 0\}$ and $\mathcal{I}_- \coloneqq \{i \in [F] : x_i < 0\}$ are non-empty sets. Also, both of $\mathcal{I}_+$ and $\mathcal{I}_-$ are closed by (2.206) and Lemma 2.6.1 stated in Subsection 2.6.3. Therefore, by Lemma 2.6.2 (ii), we obtain $\mathcal{R}_F \subseteq \mathcal{I}_+$ and $\mathcal{R}_F \subseteq \mathcal{I}_-$, which conclude $\mathcal{R}_F \subseteq \mathcal{I}_+ \cap \mathcal{I}_- = \emptyset$ as desired.

(Proof of (ii)): We show $\mathcal{R}_F = \mathcal{I}_+ \coloneqq \{i \in [F] : \pi_i(F) > 0\}$.

($\mathcal{R}_F \subseteq \mathcal{I}_+$) By (2.57), the set $\mathcal{I}_+$ is not empty. Also, by (2.56) and Lemma 2.6.1 stated in Subsection 2.6.3, the set $\mathcal{I}_+$ is closed. Hence, we obtain $\mathcal{R}_F \subseteq \mathcal{I}_+$ by Lemma 2.6.2 (ii).

($\mathcal{R}_F \supseteq \mathcal{I}_+$) Since $\mathcal{R}_F$ is closed by Lemma 2.6.2 (i), we see from Lemma 2.6.3 (ii) that the unique stationary distribution $\boldsymbol{\pi}(F)$ satisfies $\pi_i(F) = 0$ for any $i \in [F] \setminus \mathcal{R}_F$. Therefore, we obtain $\mathcal{R}_F \supseteq \mathcal{I}_+$. □

2.6.6 Proof of Lemma 2.5.4

The proof of Lemma 2.5.4 relies on Lemmas 2.6.2 and 2.6.3 stated in Subsection 2.6.5.

Proof of Lemma 2.5.4. Since $\mathcal{R}_F$ is closed by Lemma 2.6.2 (i), we see from Lemma 2.6.3 (i) that there exist $\bar{F}(\bar{f}, \bar{\tau}) \in \mathscr{F}$ and an injective homomorphism $\varphi : [\bar{F}] \to [F]$ from F' to F such that $\varphi([\bar{F}]) = \mathcal{R}_F$. Now, it suffices to show $\bar{F} \in \mathscr{F}_{\text{irr}}$.

For any $i, j \in [\bar{F}]$, there exists $\boldsymbol{x} \in \mathcal{S}^*$ such that

$$\tau^*_{\varphi(i)}(\boldsymbol{x}) = \varphi(j) \tag{2.208}$$

by $\varphi(j) \in \varphi([\bar{F}]) = \mathcal{R}_F$. Thus, for any $i, j \in [F]$, we have

$$\bar{\tau}_i^*(\boldsymbol{x}) = \varphi^{-1}(\varphi(\bar{\tau}_i^*(\boldsymbol{x}))) \overset{\text{(A)}}{=} \varphi^{-1}(\tau_{\varphi(i)}^*(\boldsymbol{x})) \overset{\text{(B)}}{=} \varphi^{-1}(\varphi(j)) = j, \tag{2.209}$$

where (A) follows from Lemma 2.5.1 (i), and (B) follows from (2.208). Therefore, $\bar{F} \in \mathscr{F}_{\text{irr}}$ holds. $\square$

Chapter 3

General Properties of k-bit Delay Decodable Optimal Codes

3.1 k-bit Delay Optimal Code-tuples

In this chapter, we consider code-tuples which achieve the optimal average codeword length in $\mathscr{F}_{\mathrm{reg}} \cap \mathscr{F}_{\mathrm{ext}} \cap \mathscr{F}_{k\text{-dec}}$ for an integer $k \geq 0$.

Definition 3.1.1. *Let $k \geq 0$ be an integer. A code-tuple $F \in \mathscr{F}_{\mathrm{reg}} \cap \mathscr{F}_{\mathrm{ext}} \cap \mathscr{F}_{k\text{-dec}}$ is said to be* k-bit delay optimal *if $L(F) \leq L(F')$ holds for any $F' \in \mathscr{F}_{\mathrm{reg}} \cap \mathscr{F}_{\mathrm{ext}} \cap \mathscr{F}_{k\text{-dec}}$. We define $\mathscr{F}_{k\text{-opt}}$ as the set of all k-bit delay optimal code-tuples, that is,*

$$\mathscr{F}_{k\text{-opt}} \coloneqq \{F \in \mathscr{F} : F \textit{ is } k\textit{-bit delay optimal.}\} = \mathop{\arg\min}_{F \in \mathscr{F}_{\mathrm{reg}} \cap \mathscr{F}_{\mathrm{ext}} \cap \mathscr{F}_{k\text{-dec}}} L(F). \tag{3.1}$$

Note that $\mathscr{F}_{k\text{-opt}}$ depends on the probability distribution μ of the source symbols, and we are now discussing on an arbitrarily fixed μ.

Example 3.1.1. *Let $(\mu(\mathrm{a}), \mu(\mathrm{b}), \mu(\mathrm{c}), \mu(\mathrm{d})) = (0.1, 0.2, 0.3, 0.4)$. Then the code-tuple $F^{(\kappa)}$ in Table 3.1 is a 2-bit delay optimal code-tuple with $L(F^{(\kappa)}) \approx 1.8667$.*

We now prove three theorems which enable us to limit the scope of code-tuples to be considered when discussing k-bit delay optimal code-tuples. In Subsections 3.1.1–3.1.3, we state the three theorems as Theorem 3.1.1–3.1.3 and give their proofs in Section 3.2–3.4, respectively.

Table 3.1: The code-tuple $F^{(\kappa)}$ is a 2-bit delay optimal code-tuple, which satisfies Theorem 3.1.1 (a)–(d) with $F = F^{(\xi)}$, where $(\mu(\mathrm{a}), \mu(\mathrm{b}), \mu(\mathrm{c}), \mu(\mathrm{d})) = (0.1, 0.2, 0.3, 0.4)$

$s \in \mathcal{S}$	$f_0^{(\xi)}$	$\tau_0^{(\xi)}$	$f_1^{(\xi)}$	$\tau_1^{(\xi)}$	$f_2^{(\xi)}$	$\tau_2^{(\xi)}$	$f_3^{(\xi)}$	$\tau_3^{(\xi)}$
a	0010	2	100	1	1100	1	010	0
b	0011	0	00	0	11	2	011	1
c	000	1	01	1	01	1	100	0
d	λ	2	1	2	10	0	1	2

$s \in \mathcal{S}$	$f_0^{(\kappa)}$	$\tau_0^{(\kappa)}$	$f_1^{(\kappa)}$	$\tau_1^{(\kappa)}$
a	100	0	1100	0
b	00	0	11	1
c	01	0	01	0
d	1	1	10	0

3.1.1 The First Theorem

The first theorem claims that for any $F \in \mathscr{F}_{\mathrm{reg}} \cap \mathscr{F}_{\mathrm{ext}} \cap \mathscr{F}_{k\text{-dec}}$, there exists $F^\dagger \in \mathscr{F}_{\mathrm{irr}} \cap \mathscr{F}_{\mathrm{ext}} \cap \mathscr{F}_{k\text{-dec}}$ such that $L(F^\dagger) \leq L(F)$ and $\mathcal{P}^k_{F^\dagger,0}, \mathcal{P}^k_{F^\dagger,1}, \ldots, \mathcal{P}^k_{F^\dagger,|F^\dagger|-1}$ are distinct. Namely, it suffices to consider only irreducible code-tuples with at most $2^{(2^k)}$ code tables to achieve a small average codeword length. In particular, it is not the case that one can achieve an arbitrarily small average codeword length by using arbitrarily many code tables. To state the theorem, we prepare the following Definition 3.1.2.

Definition 3.1.2. *For an integer $k \geq 0$ and $F \in \mathscr{F}$, we define $\mathscr{P}^k_F$ as*

$$\mathscr{P}^k_F \coloneqq \{\mathcal{P}^k_{F,i} : i \in [F]\}. \tag{3.2}$$

Example 3.1.2. *For $F^{(\gamma)}$ in Table 2.1, we have*

$$\mathscr{P}^0_{F^{(\gamma)}} = \{\{\lambda\}\}, \quad \mathscr{P}^1_{F^{(\gamma)}} = \{\{0, 1\}, \{1\}\}, \tag{3.3}$$

$$\mathscr{P}^2_{F^{(\gamma)}} = \{\{01, 10\}, \{00, 01, 10\}, \{11\}\}. \tag{3.4}$$

The following Lemma 3.1.1 holds by Lemma 2.5.1 (ii).

Lemma 3.1.1. *For any integer $k \geq 0$, $F \in \mathscr{F}_{\mathrm{reg}}$, and an irreducible part $\bar{F}$ of F, we have $\mathscr{P}^k_{\bar{F}} \subseteq \mathscr{P}^k_F$.*

Using Definition 3.1.2, we state the desired theorem as follows.

Theorem 3.1.1. *For any integer $k \geq 0$ and $F \in \mathscr{F}_{\mathrm{reg}} \cap \mathscr{F}_{\mathrm{ext}} \cap \mathscr{F}_{k\text{-dec}}$, there exists $F^\dagger \in \mathscr{F}$ satisfying the following conditions (a)–(d).*

(a) $F^\dagger \in \mathscr{F}_{\mathrm{irr}} \cap \mathscr{F}_{\mathrm{ext}} \cap \mathscr{F}_{k\text{-dec}}$.

(b) $L(F^\dagger) \leq L(F)$.

(c) $\mathscr{P}^k_{F^\dagger} \subseteq \mathscr{P}^k_F$.

(d) $|\mathscr{P}^k_{F^\dagger}| = |F^\dagger|$.

Note that $\mathcal{P}^k_{F,0}, \mathcal{P}^k_{F,1}, \ldots, \mathcal{P}^k_{F,|F|-1}$ are distinct if and only if $|\mathscr{P}^k_F| = |F|$.

Example 3.1.3. *Let $(\mu(\mathrm{a}), \mu(\mathrm{b}), \mu(\mathrm{c}), \mu(\mathrm{d})) = (0.1, 0.2, 0.3, 0.4)$ and $F := F^{(\xi)}$ in Table 3.1. Then we have $F \in \mathscr{F}_{\mathrm{reg}} \cap \mathscr{F}_{\mathrm{ext}} \cap \mathscr{F}_{2\text{-dec}}$, $L(F) \approx 1.98644$, and $\mathscr{P}^2_F = \{\{00, 01, 10, 11\}, \{01, 10, 11\}\}$. The code-tuple $F^\dagger := F^{(\kappa)}$ in Table 3.1 satisfies Theorem 3.1.1 (a)–(d) because $\mathcal{R}_{F^\dagger} = \{0, 1\} = [F^\dagger]$, $L(F^\dagger) \approx 1.8667 \leq L(F)$, and $\mathscr{P}^2_{F^\dagger} = \{\{00, 01, 10, 11\}, \{01, 10, 11\}\}$.*

Example 3.1.4. *We confirm that Theorem 3.1.1 holds for $k = 0$. Choose $F \in \mathscr{F}_{\mathrm{reg}} \cap \mathscr{F}_{\mathrm{ext}} \cap \mathscr{F}_{0\text{-dec}}$ arbitrarily and define $F^\dagger(f^\dagger, \tau^\dagger) \in \mathscr{F}^{(1)}$ as*

$$f^\dagger_0(s) = f_p(s), \tag{3.5}$$

$$\tau^\dagger_0(s) = 0 \tag{3.6}$$

for $s \in \mathcal{S}$, where

$$p \in \arg\min_{i \in [F]} L_i(F). \tag{3.7}$$

Namely, $F^\dagger$ is the 1-code-tuple consisting of the most efficient code table of F.

We can see that $F^\dagger$ satisfies Theorem 3.1.1 (a)–(d) as follows.

(a) *We obtain $F^\dagger \in \mathscr{F}_{\mathrm{irr}}$ directly from $|F^\dagger| = 1$. By $F \in \mathscr{F}_{0\text{-dec}}$ and Lemma 2.2.7, all code tables of F are prefix-free. In particular, $f^\dagger_0 = f_p$ is prefix-free and thus $F^\dagger \in \mathscr{F}_{0\text{-dec}}$. Moreover, since $f^\dagger_0$ is prefix-free and $|\mathcal{S}| \geq 2$, we have $f^\dagger_0(s) \neq \lambda$ for some $s \in \mathcal{S}$, which shows $F^\dagger \in \mathscr{F}_{\mathrm{ext}}$.*

(b) We have

$$L(F^\dagger) = L_0(F^\dagger) \overset{\text{(A)}}{=} L_p(F) = \sum_{i \in [F]} \pi_i(F) L_p(F) \overset{\text{(B)}}{\leq} \sum_{i \in [F]} \pi_i(F) L_i(F) = L(F), \tag{3.8}$$

where (A) follows from (3.5), and (B) follows from (3.7).

(c) By $\mathscr{P}^0_{F^\dagger} = \{\{\lambda\}\} = \mathscr{P}^0_F$.

(d) By $|\mathscr{P}^0_{F^\dagger}| = |\{\{\lambda\}\}| = 1 = |F^\dagger|$.

As a consequence of Theorem 3.1.1, we can prove the existence of a k-bit delay optimal code-tuple as the following Lemma 3.1.2 which proof is relegated to Subsection 3.5.1.

Lemma 3.1.2. *For any integer $k \geq 0$, there exists a k-bit delay optimal code-tuple, equivalently, $\mathscr{F}_{k\text{-opt}} \neq \emptyset$.*

3.1.2 The Second Theorem

The second theorem gives a necessary condition for $F \in \mathscr{F}_{\text{reg}} \cap \mathscr{F}_{\text{ext}} \cap \mathscr{F}_{k\text{-dec}}$ to be k-bit delay optimal. Recall that every internal node in a code-tree of Huffman code has two child nodes because of its optimality. This leads to that any bit sequence is a prefix of the codeword sequence of some source sequence. More formally,

$$^\forall \boldsymbol{b} \in \mathcal{C}^*; {}^\exists \boldsymbol{x} \in \mathcal{S}^*; f_{\text{Huff}}(\boldsymbol{x}) \succeq \boldsymbol{b}, \tag{3.9}$$

where $f_{\text{Huff}}(\boldsymbol{x})$ is the codeword sequence of $\boldsymbol{x}$ with the Huffman code. The following Theorem 3.1.2 is a generalization of this property of Huffman codes to k-bit delay decodable code-tuples for $k \geq 0$.

Theorem 3.1.2. *For any integer $k \geq 0$, $F \in \mathscr{F}_{k\text{-opt}}$, $i \in \mathcal{R}_F$, and $\boldsymbol{b} \in \mathcal{C}^{\geq k}$, if $[\boldsymbol{b}]_k \in \mathcal{P}^k_{F,i}$, then $\boldsymbol{b} \in \mathcal{P}^*_{F,i}$.*

Remark 3.1.1. *A Huffman code is represented by a 0-bit delay decodable 1-code-tuple $F \in \mathscr{F}^{(1)} \cap \mathscr{F}_{0\text{-dec}}$. We have $F \in \mathscr{F}_{0\text{-opt}}$ by the optimality of Huffman codes. Applying Theorem 3.1.2 to F with $k = 0$, we obtain*

$$^\forall \boldsymbol{b} \in \mathcal{C}^*; \boldsymbol{b} \in \mathcal{P}^*_{F,0}, \tag{3.10}$$

which is equivalent to (3.9), and thus Theorem 3.1.2 is indeed a generalization of the property (3.9) of Huffman codes.

Table 3.2: An example of $\boldsymbol{x} \in \mathcal{S}^*$ such that $f_i^*(\boldsymbol{x}) \succeq \boldsymbol{b}$, where $F(f, \tau) := F^{(\kappa)}$ in Table 3.1, $i \in \{0, 1\}$, and $\boldsymbol{b} \in \mathcal{C}^3$

i \ $\boldsymbol{b}$	000	001	010	011	100	101	110	111
0	bb	ba	cb	ca	a	dc	dd	db
1	-	-	cb	ca	db	da	a	ba

Example 3.1.5. *For $F(f, \tau) := F^{(\kappa)}$ in Table 3.1, we have $F \in \mathscr{F}_{2\text{-opt}}$ for $(\mu(\mathrm{a}), \mu(\mathrm{b}), \mu(\mathrm{c}), \mu(\mathrm{d})) = (0.1, 0.2, 0.3, 0.4)$ (cf. Example 3.1.1). Theorem 3.1.2 claims that for any $i \in \mathcal{R}_F = \{0, 1\}$ and $\boldsymbol{b} \in \mathcal{C}^{\geq 2}$ such that $b_1 b_2 \in \mathcal{P}_{F,i}^2$, it holds that $\boldsymbol{b} \in \mathcal{P}_{F,i}^*$, that is, there exists $\boldsymbol{x} \in \mathcal{S}^*$ such that $f_i^*(\boldsymbol{x}) \succeq \boldsymbol{b}$.*

For $i \in \{0, 1\}$ and $\boldsymbol{b} \in \mathcal{C}^3$ such that $b_1 b_2 \in \mathcal{P}_{F,i}^2$, Table 3.2 shows an example of $\boldsymbol{x} \in \mathcal{S}^*$ such that $f_i^*(\boldsymbol{x}) \succeq \boldsymbol{b}$. For example, we have $f_0^*(\mathrm{ca}) \succeq 011$ and $f_1^*(\mathrm{ba}) \succeq 111$. Note that $b_1 b_2 \in \mathcal{P}_{F,i}^2$ does not hold for $(i, \boldsymbol{b}) = (1, 000)$ and $(i, \boldsymbol{b}) = (1, 001)$.

3.1.3 The Third Theorem

The third theorem enables us to assume without loss of generality that a k-bit delay optimal code-tuple F satisfies $\mathcal{P}_{F,i}^1 = \{0, 1\}$ for any $i \in [F]$, that is, F belongs to the class $\mathscr{F}_{\text{fork}}$ defined as follows.

Definition 3.1.3. *We define $\mathscr{F}_{\text{fork}}$ as*

$$\mathscr{F}_{\text{fork}} := \{F \in \mathscr{F} : {}^{\forall} i \in [F], \mathcal{P}_{F,i}^1 = \{0, 1\}\}, \tag{3.11}$$

that is, $\mathscr{F}_{\text{fork}}$ is the set of all code-tuples F such that $\mathcal{P}_{F,0} = \mathcal{P}_{F,1} = \cdots = \mathcal{P}_{F,|F|-1} = \{0, 1\}$.

Theorem 3.1.3. *For any integer $k \geq 0$ and $F \in \mathscr{F}_{\text{reg}} \cap \mathscr{F}_{\text{ext}} \cap \mathscr{F}_{k\text{-dec}}$, there exists $F' \in \mathscr{F}_{\text{reg}} \cap \mathscr{F}_{\text{ext}} \cap \mathscr{F}_{k\text{-dec}} \cap \mathscr{F}_{\text{fork}}$ such that $L(F') = L(F)$.*

Corollary 3.1.1. *For any integer $k \geq 0$, we have $\mathscr{F}_{k\text{-opt}} \cap \mathscr{F}_{\text{fork}} \neq \emptyset$.*

3.2 Proof of Theorem 3.1.1

As a preparation for the proof of Theorem 3.1.1, we state the following Lemmas 3.2.1–3.2.4. See Subsection 3.5.2–3.5.4 for the proofs of Lemmas 3.2.1, 3.2.2, and 3.2.4, respectively.

Lemma 3.2.1. *Let $k \geq 0$ be an integer and let $F(f, \tau)$ and $F'(f', \tau')$ be code-tuples such that $|F| = |F'|$. Assume that the following conditions (a) and (b) hold.*

(a) $f_i(s) = f'_i(s)$ *for any* $i \in [F]$ *and* $s \in \mathcal{S}$.

(b) $\mathcal{P}^k_{F,\tau_i(s)} = \mathcal{P}^k_{F,\tau'_i(s)}$ *for any* $i \in [F]$ *and* $s \in \mathcal{S}$.

Then the following statements (i)–(iii) hold.

(i) For any $i \in [F']$ *and* $\boldsymbol{b} \in \mathcal{C}^*$, *we have* $\mathcal{P}^k_{F,i}(\boldsymbol{b}) = \mathcal{P}^k_{F',i}(\boldsymbol{b})$ *and* $\bar{\mathcal{P}}^k_{F,i}(\boldsymbol{b}) = \bar{\mathcal{P}}^k_{F',i}(\boldsymbol{b})$.

(ii) If $F \in \mathscr{F}_{\mathrm{ext}}$, *then* $F' \in \mathscr{F}_{\mathrm{ext}}$.

(iii) If $F \in \mathscr{F}_{k\text{-dec}}$, *then* $F' \in \mathscr{F}_{k\text{-dec}}$.

Lemma 3.2.2. *For any* $F(f, \tau) \in \mathscr{F}_{\mathrm{irr}}$, $\mathcal{I} \subseteq [F]$, *and* $p \in \mathcal{I}$, *the code-tuple* $F'(f', \tau') \in \mathscr{F}^{(|F|)}$ *defined as (3.12) and (3.13) satisfies* $F' \in \mathscr{F}_{\mathrm{reg}}$:

$$f'_i(s) = f_i(s), \tag{3.12}$$

$$\tau'_i(s) = \begin{cases} p & \text{if } \tau_i(s) \in \mathcal{I}, \\ \tau_i(s) & \text{if } \tau_i(s) \notin \mathcal{I} \end{cases} \tag{3.13}$$

for $i \in [F']$ *and* $s \in \mathcal{S}$.

Lemma 3.2.3. *For any* $F \in \mathscr{F}$, *there exists* $(h_0, h_1, \ldots, h_{|F|-1}) \in \mathbb{R}^{|F|}$ *satisfying*

$$^\forall i \in [F]; L(F) = L_i(F) + \sum_{j \in [F]} (h_j - h_i) Q_{i,j}(F). \tag{3.14}$$

See [24, Sec. 8.2] for proof of Lemma 3.2.3. The vector h called "bias" defined as [24, (8.2.2)] satisfies (3.14) of this thesis. This fact is shown as [24, (8.2.12)] in [24, Theorem 8.2.6], where g, r, and P in [24, (8.2.12)] correspond to the notations of this thesis as follows:

$$g = \begin{bmatrix} L(F) \\ L(F) \\ \vdots \\ L(F) \end{bmatrix}, \quad r = \begin{bmatrix} L_0(F) \\ L_1(F) \\ \vdots \\ L_{|F|-1}(F) \end{bmatrix}, \quad P = Q(F). \tag{3.15}$$

A real vector $(h_0, h_1, \ldots, h_{|F|-1})$ satisfying (3.14) is not unique. We refer to arbitrarily chosen one of them as $h(F) = (h_0(F), h_1(F), \ldots, h_{|F|-1}(F))$.

Lemma 3.2.4. *For any $F(f,\tau), F'(f',\tau') \in \mathscr{F}_{\rm reg}$ such that $|F| = |F'|$, if the following conditions (a) and (b) hold, then $L(F') \le L(F)$.*

(a) $L_i(F) = L_i(F')$ for any $i \in [F]$.

(b) $h_{\tau_i(s)}(F) \ge h_{\tau'_i(s)}(F)$ for any $i \in [F]$ and $s \in \mathcal{S}$.

Using these lemmas, we now prove Theorem 3.1.1.

Proof of Theorem 3.1.1. We fix an integer $k \ge 0$ arbitrarily and prove Theorem 3.1.1 by induction for $|F|$. For the base case $|F| = 1$, the code-tuple $F^\dagger := F$ satisfies (a)–(d) of Theorem 3.1.1 as desired. We now consider the induction step for $|F| \ge 2$.

We consider an irreducible part $\bar{F}(\bar{f},\bar{\tau})$ of F. By Lemmas 2.5.3 and 3.1.1, the following statements (ā)–(c̄) hold (cf. (a)–(c) of Theorem 3.1.1).

(ā) $\bar{F} \in \mathscr{F}_{\rm irr} \cap \mathscr{F}_{\rm ext} \cap \mathscr{F}_{k\text{-dec}}$.

(b̄) $L(\bar{F}) = L(F)$.

(c̄) $\mathscr{P}^k_{\bar{F}} \subseteq \mathscr{P}^k_F$.

Therefore, if $|\mathscr{P}^k_{\bar{F}}| = |\bar{F}|$, then $F^\dagger := \bar{F}$ satisfies (a)–(d) of Theorem 3.1.1 as desired. Thus, we now assume $|\mathscr{P}^k_{\bar{F}}| < |\bar{F}|$. Then we can choose $i', j' \in [\bar{F}]$ such that $i' \ne j'$ and $\mathcal{P}^k_{\bar{F},i'} = \mathcal{P}^k_{\bar{F},j'}$ by pigeonhole principle. We define $F'(f',\tau') \in \mathscr{F}^{(|\bar{F}|)}$ as

$$f'_i(s) = \bar{f}_i(s), \tag{3.16}$$

$$\tau'_i(s) = \begin{cases} p & \text{if } \bar{\tau}_i(s) \in \mathcal{I}, \\ \bar{\tau}_i(s) & \text{if } \bar{\tau}_i(s) \notin \mathcal{I} \end{cases} \tag{3.17}$$

for $i \in [F']$ and $s \in \mathcal{S}$, where

$$\mathcal{I} := \{i \in [\bar{F}] : \mathcal{P}^k_{\bar{F},i} = \mathcal{P}^k_{\bar{F},i'} (= \mathcal{P}^k_{\bar{F},j'})\} \tag{3.18}$$

and we choose

$$p \in \arg\min_{i \in \mathcal{I}} h_i(\bar{F}) \tag{3.19}$$

arbitrarily.

Then we obtain $F' \in \mathscr{F}_{\mathrm{reg}}$ by applying Lemma 3.2.2 since $\bar{F} \in \mathscr{F}_{\mathrm{irr}}$. Also, we obtain $F' \in \mathscr{F}_{\mathrm{ext}} \cap \mathscr{F}_{k\text{-dec}}$ and

$$\mathscr{P}^k_{F'} = \mathscr{P}^k_{\bar{F}} \tag{3.20}$$

for any $i \in [F']$ by applying Lemma 3.2.1 (i)–(iii) since $\bar{f}_i(s) = f'_i(s)$ and $\mathcal{P}^k_{\bar{F},\bar{\tau}_i(s)} = \mathcal{P}^k_{\bar{F},\tau'_i(s)}$ for any $i \in [\bar{F}]$ and $s \in \mathcal{S}$ by (3.16) and (3.17). Moreover, we can see

$$L(F') \leq L(\bar{F}) \tag{3.21}$$

by applying Lemma 3.2.4 because F' satisfies (a) (resp. (b)) of Lemma 3.2.4 by (3.16) (resp. (3.17)–(3.19)).

Since $|\mathcal{I}| \geq |\{i', j'\}| \geq 2$, we have $\mathcal{I} \setminus \{p\} \neq \emptyset$. Also, for any $i \in \mathcal{I} \setminus \{p\}$, we have $i \notin \mathcal{R}_{F'}$ since for any $j \in [F'] \setminus \{i\}$, there exists no $\boldsymbol{x} \in \mathcal{S}^*$ such that $\tau'^*_j(\boldsymbol{x}) = i$ by (3.17). Therefore, we have

$$\mathcal{R}_{F'} \subsetneq [F']. \tag{3.22}$$

For an irreducible part $\bar{F}'$ of F', we have

$$|\bar{F}'| = |\mathcal{R}_{F'}| \overset{\text{(A)}}{<} |F'| = |\bar{F}| = |\mathcal{R}_F| \leq |F|, \tag{3.23}$$

where (A) follows from (3.22). Therefore, by applying the induction hypothesis to $\bar{F}'$, we can see that there exists $F^\dagger \in \mathscr{F}$ satisfying the following conditions (a†)–(d†).

(a†) $F^\dagger \in \mathscr{F}_{\mathrm{irr}} \cap \mathscr{F}_{\mathrm{ext}} \cap \mathscr{F}_{k\text{-dec}}$.

(b†) $L(F^\dagger) \leq L(\bar{F}')$.

(c†) $\mathscr{P}^k_{F^\dagger} \subseteq \mathscr{P}^k_{\bar{F}'}$.

(d†) $|\mathscr{P}^k_{F^\dagger}| = |F^\dagger|$.

We can see that $F^\dagger$ is a desired code-tuple, that is, $F^\dagger$ satisfies (a)–(d) of Theorem 3.1.1 as follows. First, (a) and (d) are directly from (a†) and (d†), respectively. We obtain (b) as follows:

$$L(F^\dagger) \overset{\text{(A)}}{\leq} L(\bar{F}') \overset{\text{(B)}}{=} L(F') \overset{\text{(C)}}{\leq} L(\bar{F}) \overset{\text{(D)}}{=} L(F), \tag{3.24}$$

where (A) follows from ($b^\dagger$), (B) follows from Lemma 2.5.3, (C) follows from (3.21), and (D) follows from Lemma 2.5.3. The condition (c) holds because

$$\mathscr{P}^k_{F^\dagger} \overset{\text{(A)}}{\subseteq} \mathscr{P}^k_{\bar{F}'} \overset{\text{(B)}}{\subseteq} \mathscr{P}^k_{F'} \overset{\text{(C)}}{=} \mathscr{P}^k_{\bar{F}} \overset{\text{(D)}}{\subseteq} \mathscr{P}^k_{F}, \tag{3.25}$$

where (A) follows from ($c^\dagger$), (B) follows from Lemma 3.1.1, (C) follows from (3.20), and (D) follows from Lemma 3.1.1. □

3.3 Proof of Theorem 3.1.2

Proof of Theorem 3.1.2. We prove by contradiction assuming that there exist $p \in \mathcal{R}_F$ and $\boldsymbol{b} = b_1 b_2 \ldots b_l \in \mathcal{C}^{\geq k}$ such that

$$\boldsymbol{b} \notin \mathcal{P}^*_{F,p}, \quad b_1 b_2 \ldots b_k \in \mathcal{P}^k_{F,p}. \tag{3.26}$$

Without loss of generality, we assume $p = |F| - 1$ and $\boldsymbol{b}$ is the shortest sequence satisfying (3.26). Because we have $l > k$ by (3.26), we have $\mathrm{pref}(\boldsymbol{b}) \succeq b_1 b_2 \ldots b_k \in \mathcal{P}^k_{F,|F|-1}$. Since $\boldsymbol{b}$ is the shortest sequence satisfying (3.26), it must hold that $\mathrm{pref}(\boldsymbol{b}) \in \mathcal{P}^*_{F,|F|-1}$. Hence, by $F \in \mathscr{F}_{\mathrm{ext}}$ and Lemma 2.3.2 (i), we have $\boldsymbol{d} = d_1 d_2 \ldots d_l := \mathrm{pref}(\boldsymbol{b})\bar{b}_l \in \mathcal{P}^*_{F,|F|-1}$. Namely, we have

$$\boldsymbol{d} \in \mathcal{P}^*_{F,|F|-1}, \quad \mathrm{pref}(\boldsymbol{d})\bar{d}_l = \boldsymbol{b} \notin \mathcal{P}^*_{F,|F|-1}. \tag{3.27}$$

We state the key idea of the proof as follows. By (3.27), whenever the decoder reads a prefix $\mathrm{pref}(\boldsymbol{d})$ of the codeword sequence, the decoder can know that the following bit is d_l without reading it. Hence, the bit d_l gives no information and is unnecessary for the k-bit delay decodability of the mapping $f^*_{|F|-1}$. We consider obtaining another code-tuple $F'' \in \mathscr{F}_{\mathrm{reg}} \cap \mathscr{F}_{\mathrm{ext}} \cap \mathscr{F}_{k\text{-dec}}$ such that $L(F'') < L(F)$ by removing this redundant bit, which leads to a contradiction to $F \in \mathscr{F}_{k\text{-opt}}$ as desired. However, naive removing a bit may impair the k-bit delay decodability of the other mappings f^*_i for $i \in [|F| - 1]$. Accordingly, we first define a code-tuple F' which is essentially equivalent to F by adding some duplicates of the code tables to F. Then by making changes to the replicated code tables instead of the original code tables, we obtain the desired F'' without affecting the k-bit delay decodability of f^*_i for $i \in [|F| - 1]$.

We define the code-tuple F' as follows. Put $L := |F|(|\boldsymbol{d}| + 1)$ and $M := |\mathcal{S}^{\leq L}|$. We number all the sequences of $\mathcal{S}^{\leq L}$ as $\boldsymbol{z}^{(0)}, \boldsymbol{z}^{(1)}, \boldsymbol{z}^{(2)}, \ldots, \boldsymbol{z}^{(M-1)}$

in any order but $\boldsymbol{z}^{(0)} := \lambda$. For $\boldsymbol{z}' \in \mathcal{S}^{\leq L}$, we define $\langle \boldsymbol{z}' \rangle := |F| - 1 + t$, where t is the integer such that $\boldsymbol{z}^{(t)} = \boldsymbol{z}'$. Note that $\langle \lambda \rangle = |F| - 1$ since $\boldsymbol{z}^{(0)} = \lambda$. We define the code-tuple $F' \in \mathscr{F}^{(|F|-1+M)}$ consisting of $f'_0, f'_1, \ldots, f'_{|F|-1}, f'_{\langle \boldsymbol{z}^{(1)} \rangle}, f'_{\langle \boldsymbol{z}^{(2)} \rangle}, \ldots, f'_{\langle \boldsymbol{z}^{(M-1)} \rangle}$ and $\tau'_0, \tau'_1, \ldots, \tau'_{|F|-1}, \tau'_{\langle \boldsymbol{z}^{(1)} \rangle}, \tau'_{\langle \boldsymbol{z}^{(2)} \rangle}, \ldots, \tau'_{\langle \boldsymbol{z}^{(M-1)} \rangle}$ as

$$f'_i(s) = \begin{cases} f_{\tau^*_{\langle \lambda \rangle}(\boldsymbol{z})}(s) & \text{if } i = \langle \boldsymbol{z} \rangle \text{ for some } \boldsymbol{z} \in \mathcal{S}^{\leq L}, \\ f_i(s) & \text{otherwise,} \end{cases} \tag{3.28}$$

$$\tau'_i(s) = \begin{cases} \langle \boldsymbol{z}s \rangle & \text{if } i = \langle \boldsymbol{z} \rangle \text{ for some } \boldsymbol{z} \in \mathcal{S}^{\leq L-1}, \\ \tau^*_{\langle \lambda \rangle}(\boldsymbol{z}s) & \text{if } i = \langle \boldsymbol{z} \rangle \text{ for some } \boldsymbol{z} \in \mathcal{S}^{L}, \\ \tau_i(s) & \text{otherwise} \end{cases} \tag{3.29}$$

for $i \in [F']$ and $s \in \mathcal{S}$. Then F' satisfies the following Lemma 3.3.1. See Subsection 3.5.5 for the proof of Lemma 3.3.1.

Lemma 3.3.1. *For any $\boldsymbol{z} \in \mathcal{S}^{\leq L}$, the following statements (i) and (ii) hold.*

(i) $\tau'^*_{\langle \lambda \rangle}(\boldsymbol{z}) = \langle \boldsymbol{z} \rangle$.

(ii) $\langle \boldsymbol{z} \rangle \in \mathcal{R}_{F'}$.

Lemma 3.3.1 (i) claims that the code table in F' used next after encoding $\boldsymbol{z} \in \mathcal{S}^{\leq L}$ starting from $f'_{\langle \lambda \rangle}$ is $f'_{\langle \boldsymbol{z} \rangle}$, which is a duplicate of the code table in F used next after encoding $\boldsymbol{z}$ starting from $f_{\langle \lambda \rangle}$. This leads to the equivalency of F and F' shown next.

We confirm that F' is equivalent to F, that is, $F' \in \mathscr{F}_{\mathrm{reg}} \cap \mathscr{F}_{\mathrm{ext}} \cap \mathscr{F}_{k\text{-dec}}$ and $L(F') = L(F)$. We obtain $F' \in \mathscr{F}_{\mathrm{reg}}$ from Lemma 3.3.1 (ii) and Lemma 2.5.2 (i). To prove $F' \in \mathscr{F}_{\mathrm{ext}} \cap \mathscr{F}_{k\text{-dec}}$ and $L(F') = L(F)$ by using Lemma 2.5.1, we show that a mapping $\varphi : [F'] \to [F]$ defined as the following (3.30) is a homomorphism:

$$\varphi(i) = \begin{cases} i & \text{if } i \in [F], \\ \tau^*_{\langle \lambda \rangle}(\boldsymbol{z}) & \text{if } i = \langle \boldsymbol{z} \rangle \text{ for some } \boldsymbol{z} \in \mathcal{S}^{\leq L} \end{cases} \tag{3.30}$$

for $i \in [F']$. The case $i = |F| - 1 = \langle \lambda \rangle$ applies to both of the first and second cases of (3.30). However, this case is consistent since $\tau^*_{\langle \lambda \rangle}(\boldsymbol{z}) = \tau^*_{\langle \lambda \rangle}(\lambda) = \langle \lambda \rangle = i$. We see that φ satisfies (2.64) directly from (3.28) and (3.30). We confirm that φ satisfies also (2.65) as follows:

$$\varphi(\tau'_i(s)) \overset{\text{(A)}}{=} \begin{cases} \varphi(\langle \boldsymbol{z}s\rangle) & \text{if } i = \langle \boldsymbol{z}\rangle \text{ for some } \boldsymbol{z} \in \mathcal{S}^{\leq L-1}, \\ \varphi(\tau^*_{\langle\lambda\rangle}(\boldsymbol{z}s)) & \text{if } i = \langle \boldsymbol{z}\rangle \text{ for some } \boldsymbol{z} \in \mathcal{S}^{L}, \\ \varphi(\tau_i(s)) & \text{otherwise,} \end{cases} \tag{3.31}$$

$$\overset{\text{(B)}}{=} \begin{cases} \tau^*_{\langle\lambda\rangle}(\boldsymbol{z}s) & \text{if } i = \langle \boldsymbol{z}\rangle \text{ for some } \boldsymbol{z} \in \mathcal{S}^{\leq L-1}, \\ \tau^*_{\langle\lambda\rangle}(\boldsymbol{z}s) & \text{if } i = \langle \boldsymbol{z}\rangle \text{ for some } \boldsymbol{z} \in \mathcal{S}^{L}, \\ \tau_i(s) & \text{otherwise,} \end{cases} \tag{3.32}$$

$$\overset{\text{(C)}}{=} \begin{cases} \tau_{\tau^*_{\langle\lambda\rangle}(\boldsymbol{z})}(s) & \text{if } i = \langle \boldsymbol{z}\rangle \text{ for some } \boldsymbol{z} \in \mathcal{S}^{\leq L}, \\ \tau_i(s) & \text{otherwise,} \end{cases} \tag{3.33}$$

$$\overset{\text{(D)}}{=} \tau_{\varphi(i)}(s), \tag{3.34}$$

where (A) follows from (3.29), (B) follows from (3.30), (C) follows from Lemma 2.1.1 (ii), and (D) follows from (3.30). Hence, by Lemma 2.5.1 (iv)–(vi), we obtain $F' \in \mathscr{F}_{\text{ext}} \cap \mathscr{F}_{k\text{-dec}}$ and $L(F') = L(F)$.

Now, we define a code-tuple $F'' \in \mathscr{F}^{(|F'|)}$ as

$$f''_i(s) = \begin{cases} f'^*_{\langle\lambda\rangle}(\boldsymbol{z})^{-1}\text{pref}(\boldsymbol{d})\boldsymbol{d}^{-1}(f'^*_{\langle\lambda\rangle}(\boldsymbol{z}s)) \\ \quad \text{if } i = \langle \boldsymbol{z}\rangle \text{ and } f'^*_{\langle\lambda\rangle}(\boldsymbol{z}) \prec \boldsymbol{d} \preceq f'^*_{\langle\lambda\rangle}(\boldsymbol{z}s) \quad \text{for some } \boldsymbol{z} \in \mathcal{S}^{\leq L}, \\ f'_i(s) \quad\quad \text{otherwise,} \end{cases} \tag{3.35}$$

$$\tau''_i(s) = \tau'_i(s) \tag{3.36}$$

for $i \in [F'']$ and $s \in \mathcal{S}$.

Intuitively, (3.35) means that F'' is obtained by removing the bit d_l from codeword sequences of F' such that $f'^*_{\langle\lambda\rangle}(\boldsymbol{z}) \succeq \boldsymbol{d}$.

Then F'' satisfies the following Lemma 3.3.2. See Subsection 3.5.6 for the proof of Lemma 3.3.2.

Lemma 3.3.2. *The following statements (i)–(iii) hold.*

(i) For any $\boldsymbol{z} \in \mathcal{S}^{\leq L}$ and $\boldsymbol{x} \in \mathcal{S}^{\leq L-|\boldsymbol{z}|}$, we have

$$f''^*_{\langle\boldsymbol{z}\rangle}(\boldsymbol{x}) = \begin{cases} f'^*_{\langle\lambda\rangle}(\boldsymbol{z})^{-1}\text{pref}(\boldsymbol{d})\boldsymbol{d}^{-1}(f'^*_{\langle\lambda\rangle}(\boldsymbol{z}\boldsymbol{x})) & \textit{if } f'^*_{\langle\lambda\rangle}(\boldsymbol{z}) \prec \boldsymbol{d} \preceq f'^*_{\langle\lambda\rangle}(\boldsymbol{z}\boldsymbol{x}), \\ f'^*_{\langle\boldsymbol{z}\rangle}(\boldsymbol{x}) & \textit{otherwise.} \end{cases} \tag{3.37}$$

(ii) For any $\boldsymbol{z} \in \mathcal{S}^{\leq L}$ *and* $s, s' \in \mathcal{S}$, *if* $f''_{\langle \boldsymbol{z} \rangle}(s) \prec f''_{\langle \boldsymbol{z} \rangle}(s')$, *then* $f'_{\langle \boldsymbol{z} \rangle}(s) \prec f'_{\langle \boldsymbol{z} \rangle}(s')$.

(iii) For any $\boldsymbol{x} \in \mathcal{S}^{\geq L}$, *we have* $|f^*_{\langle \lambda \rangle}(\boldsymbol{x})| = |f'^*_{\langle \lambda \rangle}(\boldsymbol{x})| \geq |\boldsymbol{d}| + 1$ *and* $|f''^*_{\langle \lambda \rangle}(\boldsymbol{x})| \geq |\boldsymbol{d}|$.

We show that $F'' \in \mathscr{F}_{\mathrm{reg}} \cap \mathscr{F}_{\mathrm{ext}} \cap \mathscr{F}_{k\text{-dec}}$ and $L(F'') < L(F')$ ($= L(F)$ as shown above), which conflicts with $F \in \mathscr{F}_{k\text{-opt}}$ and completes the proof of Theorem 3.1.2.

(Proof of $F'' \in \mathscr{F}_{\mathrm{reg}}$): From $F' \in \mathscr{F}_{\mathrm{reg}}$ and (3.36).

(Proof of $F'' \in \mathscr{F}_{\mathrm{ext}}$): Choose $j \in [F'']$ arbitrarily. Since $\langle \lambda \rangle \in \mathcal{R}_{F'} = \mathcal{R}_{F''}$ by Lemma 3.3.1 (ii) and (3.36), there exists $\boldsymbol{x} \in \mathcal{S}^*$ such that

$$\tau''^*_j(\boldsymbol{x}) = \langle \lambda \rangle. \tag{3.38}$$

Also, we can choose $\boldsymbol{x}' \in \mathcal{S}^L$ such that

$$f'^*_{\langle \lambda \rangle}(\boldsymbol{x}') \succeq \boldsymbol{d} \tag{3.39}$$

by Lemma 3.3.2 (iii). We have

$$\begin{aligned}
|f''^*_j(\boldsymbol{x}\boldsymbol{x}')| &\overset{\text{(A)}}{=} |f''^*_j(\boldsymbol{x})| + |f''^*_{\tau''^*_j(\boldsymbol{x})}(\boldsymbol{x}')| && (3.40)\\
&\geq |f''^*_{\tau''^*_j(\boldsymbol{x})}(\boldsymbol{x}')| && (3.41)\\
&\overset{\text{(B)}}{=} |f''^*_{\langle \lambda \rangle}(\boldsymbol{x}')| && (3.42)\\
&\overset{\text{(C)}}{=} |f'^*_{\langle \lambda \rangle}(\lambda)^{-1}\mathrm{pref}(\boldsymbol{d})\boldsymbol{d}^{-1}f'^*_{\langle \lambda \rangle}(\boldsymbol{x}')| && (3.43)\\
&= |f'^*_{\langle \lambda \rangle}(\boldsymbol{x}')| - 1 && (3.44)\\
&\overset{\text{(D)}}{\geq} |\boldsymbol{d}| && (3.45)\\
&\geq 1, && (3.46)
\end{aligned}$$

where (A) follows from Lemma 2.1.1 (i), (B) follows from (3.38), (C) follows from (3.39) and the first case of (3.37), and (D) follows from Lemma 3.3.2 (iii). Hence, by (2.16), $\mathcal{P}^1_{F'',j} \neq \emptyset$ holds for any $j \in [F'']$, which leads to $F'' \in \mathscr{F}_{\mathrm{ext}}$ as desired.

(Proof of $L(F'') < L(F')$): For any $i \in [F'']$ and $s \in \mathcal{S}$, we have $|f''_i(s)| \leq |f'_i(s)|$ by (3.35). Hence, for any $i \in [F'']$, we have

$$\pi_i(F')L_i(F'') \leq \pi_i(F')L_i(F'). \tag{3.47}$$

By Lemma 3.3.2 (iii), we can choose $\boldsymbol{x} = x_1 x_2 \ldots x_L \in \mathcal{S}^L$ such that $f'^{*}_{\langle\lambda\rangle}(\boldsymbol{x}) \succeq \boldsymbol{d}$. Since $f'^{*}_{\langle\lambda\rangle}(\lambda) \prec \boldsymbol{d} \preceq f'^{*}_{\langle\lambda\rangle}(\boldsymbol{x})$, there exists exactly one integer r such that

$$f'^{*}_{\langle\lambda\rangle}(x_1 x_2 \ldots x_{r-1}) \prec \boldsymbol{d} \preceq f'^{*}_{\langle\lambda\rangle}(x_1 x_2 \ldots x_r), \tag{3.48}$$

which leads to

$$|f''_{\langle\boldsymbol{z}\rangle}(x_r)| \overset{\text{(A)}}{=} |f'^{*}_{\langle\lambda\rangle}(\boldsymbol{z})^{-1}\mathrm{pref}(\boldsymbol{d})\boldsymbol{d}^{-1}(f'^{*}_{\langle\lambda\rangle}(\boldsymbol{z}x_r))| = |f'_{\langle\boldsymbol{z}\rangle}(x_r)| - 1 < |f'_{\langle\boldsymbol{z}\rangle}(x_r)|, \tag{3.49}$$

where $\boldsymbol{z} \coloneqq x_1 x_2 \ldots x_{r-1}$, and (A) follows from (3.48) and the first case of (3.35). This leads to

$$\pi_{\langle\boldsymbol{z}\rangle}(F')L_{\langle\boldsymbol{z}\rangle}(F'') < \pi_{\langle\boldsymbol{z}\rangle}(F')L_{\langle\boldsymbol{z}\rangle}(F') \tag{3.50}$$

because $\pi_{\langle\boldsymbol{z}\rangle}(F') > 0$ by Lemma 3.3.1 (ii) and Lemma 2.5.2 (ii).

Hence, we have

$$\begin{aligned}
L(F'') &= \sum_{i \in [F'']} \pi_i(F'')L_i(F'') && (3.51)\\
&\overset{\text{(A)}}{=} \sum_{i \in [F'']} \pi_i(F')L_i(F'') && (3.52)\\
&= \sum_{i \in [F''] \setminus \{\langle\boldsymbol{z}\rangle\}} \pi_i(F')L_i(F'') + \pi_{\langle\boldsymbol{z}\rangle}(F')L_{\langle\boldsymbol{z}\rangle}(F'') && (3.53)\\
&\overset{\text{(B)}}{\leq} \sum_{i \in [F''] \setminus \{\langle\boldsymbol{z}\rangle\}} \pi_i(F')L_i(F') + \pi_{\langle\boldsymbol{z}\rangle}(F')L_{\langle\boldsymbol{z}\rangle}(F'') && (3.54)\\
&\overset{\text{(C)}}{<} \sum_{i \in [F''] \setminus \{\langle\boldsymbol{z}\rangle\}} \pi_i(F')L_i(F') + \pi_{\langle\boldsymbol{z}\rangle}(F')L_{\langle\boldsymbol{z}\rangle}(F') && (3.55)\\
&= \sum_{i \in [F']} \pi_i(F')L_i(F') && (3.56)\\
&= L(F') && (3.57)
\end{aligned}$$

as desired, where (A) follows from (3.36), (B) follows from (3.47), and (C) follows from (3.50).

(Proof of $F'' \in \mathscr{F}_{k\text{-dec}}$): To prove $F'' \in \mathscr{F}_{k\text{-dec}}$, we use the following Lemma 3.3.3, where $\mathcal{J} \coloneqq ([F'] \setminus \langle\lambda\rangle) \cup \{\langle\boldsymbol{z}\rangle : \boldsymbol{z} \in \mathcal{S}^L\} = [F'] \setminus \{\langle\boldsymbol{z}\rangle : \boldsymbol{z} \in \mathcal{S}^{\leq L-1}\}$. See Subsection 3.5.7 for the proof of Lemma 3.3.3.

Lemma 3.3.3. *The following statements (i)–(iii) hold.*

(i) For any $\boldsymbol{x} \in \mathcal{S}^*$ and $\boldsymbol{c} \in \mathcal{C}^{\leq k}$, if $f''^*_{\langle\lambda\rangle}(\boldsymbol{x}) \succeq \boldsymbol{c}$, then $f'^*_{\langle\lambda\rangle}(\boldsymbol{x}) \succeq \boldsymbol{c}$. Therefore, we have $\mathcal{P}^k_{F',\langle\lambda\rangle} \supseteq \mathcal{P}^k_{F'',\langle\lambda\rangle}$ by (2.16).

(ii) For any $i \in \mathcal{J}$ and $s \in \mathcal{S}$, we have $f''_i(s) = f'_i(s)$.

(iii) For any $i \in \mathcal{J}$ and $\boldsymbol{b} \in \mathcal{C}^*$, we have $\mathcal{P}^k_{F'',i}(\boldsymbol{b}) \subseteq \mathcal{P}^k_{F',i}(\boldsymbol{b})$ and $\bar{\mathcal{P}}^k_{F'',i}(\boldsymbol{b}) \subseteq \bar{\mathcal{P}}^k_{F',i}(\boldsymbol{b})$.

Also, for $\boldsymbol{z} \in \mathcal{S}^*$, we define a mapping $\psi_{\boldsymbol{z}} : \mathcal{C}^* \to \mathcal{C}^*$ as

$$\psi_{\boldsymbol{z}}(\boldsymbol{b}) = \begin{cases} f'^*_{\langle\lambda\rangle}(\boldsymbol{z})^{-1}\boldsymbol{d}\mathrm{pref}(\boldsymbol{d})^{-1}(f'^*_{\langle\lambda\rangle}(\boldsymbol{z})\boldsymbol{b}) & \text{if } f'^*_{\langle\lambda\rangle}(\boldsymbol{z}) \preceq \mathrm{pref}(\boldsymbol{d}) \prec f'^*_{\langle\lambda\rangle}(\boldsymbol{z})\boldsymbol{b}, \\ \boldsymbol{b} & \text{otherwise} \end{cases} \tag{3.58}$$

for $\boldsymbol{b} \in \mathcal{C}^*$. Then $\psi_{\boldsymbol{z}}$ satisfies the following Lemma 3.3.4.

Lemma 3.3.4. *The following statements (i)–(iii) hold.*

(i) For any $\boldsymbol{z} \in \mathcal{S}^*$ and $\boldsymbol{b}, \boldsymbol{b}' \in \mathcal{C}^*$, if $\boldsymbol{b} \preceq \boldsymbol{b}'$, then $\psi_{\boldsymbol{z}}(\boldsymbol{b}) \preceq \psi_{\boldsymbol{z}}(\boldsymbol{b}')$.

(ii) For any $\boldsymbol{z} \in \mathcal{S}^{\leq L}$, $\boldsymbol{x} \in \mathcal{S}^{\leq L-|\boldsymbol{z}|}$, and $\boldsymbol{c} \in \mathcal{C}^*$, we have

$$\psi_{\boldsymbol{z}}(f''^*_{\langle\boldsymbol{z}\rangle}(\boldsymbol{x})\boldsymbol{c}) = \begin{cases} \mathrm{pref}(f'^*_{\langle\boldsymbol{z}\rangle}(\boldsymbol{x})) & \text{if } f'^*_{\langle\lambda\rangle}(\boldsymbol{z}) \prec f'^*_{\langle\lambda\rangle}(\boldsymbol{z}\boldsymbol{x}) = \boldsymbol{d}, \boldsymbol{c} = \lambda, \\ f'^*_{\langle\boldsymbol{z}\rangle}(\boldsymbol{x})\psi_{\boldsymbol{z}\boldsymbol{x}}(\boldsymbol{c}) & \text{otherwise.} \end{cases} \tag{3.59}$$

(iii) For any $\boldsymbol{z} \in \mathcal{S}^L$ and $\boldsymbol{b} \in \mathcal{C}^*$, we have $\psi_{\boldsymbol{z}}(\boldsymbol{b}) = \boldsymbol{b}$.

See Subsection 3.5.8 for the proof of Lemma 3.3.4.

By Lemma 3.3.4 (ii) with $\boldsymbol{c} = \lambda$, it holds that $\psi_{\boldsymbol{z}}(f''^*_{\langle\boldsymbol{z}\rangle}(\boldsymbol{x})) = f'^*_{\langle\boldsymbol{z}\rangle}(\boldsymbol{x})$ in most cases. Thus, we can intuitively interpret the mapping $\psi_{\boldsymbol{z}}$ as a kind of an inverse transformation of (3.37). We prove k-bit delay decodability of F'' later by attributing it to k-bit delay decodability of F' using $\psi_{\boldsymbol{z}}$.

Now we prove $F'' \in \mathscr{F}_{k\text{-dec}}$. We first show that F'' satisfies Definition 2.2.3 (a). Namely, we show that $\mathcal{P}^k_{F'',\tau'_i(s)} \cap \bar{\mathcal{P}}^k_{F'',i}(f''_i(s)) = \emptyset$ for any $i \in [F'']$ and $s \in \mathcal{S}$ dividing into the following two cases: the case $i \in \mathcal{J}$ and the case $i \in [F''] \setminus \mathcal{J}$.

- The case $i \in \mathcal{J}$: Then for any $i \in \mathcal{J}$ and $s \in \mathcal{S}$, we have

$$\mathcal{P}^k_{F'',\tau''_i(s)} \cap \bar{\mathcal{P}}^k_{F'',i}(f''_i(s)) \overset{\text{(A)}}{\subseteq} \mathcal{P}^k_{F',\tau''_i(s)} \cap \bar{\mathcal{P}}^k_{F',i}(f''_i(s)) \tag{3.60}$$

$$\overset{\text{(B)}}{=} \mathcal{P}^k_{F',\tau'_i(s)} \cap \bar{\mathcal{P}}^k_{F',i}(f'_i(s)) \tag{3.61}$$

$$\overset{\text{(C)}}{=} \emptyset, \tag{3.62}$$

where (A) follows from Lemma 3.3.3 (i) (iii) since $\tau''_i(s) \in [F]$, (B) follows from Lemma 3.3.3 (ii) and (3.36), and (C) follows from $F' \in \mathscr{F}_{k\text{-dec}}$.

- The case $i \in [F''] \setminus \mathcal{J}$: We prove by contradiction assuming that there exist $\boldsymbol{z} \in \mathcal{S}^{\leq L-1}$, $s \in \mathcal{S}$, and $\boldsymbol{c} \in \bar{\mathcal{P}}^k_{F'',\langle \boldsymbol{z} \rangle}(f''_{\langle \boldsymbol{z} \rangle}(s)) \cap \mathcal{P}^k_{F'',\langle \boldsymbol{z}s \rangle}$. By $\boldsymbol{c} \in \bar{\mathcal{P}}^k_{F'',\langle \boldsymbol{z} \rangle}(f''_{\langle \boldsymbol{z} \rangle}(s))$ and (2.13), there exist $\boldsymbol{x} \in \mathcal{S}^{L-|\boldsymbol{z}|}$ and $\boldsymbol{y} \in \mathcal{S}^*$ such that

$$f''^*_{\langle \boldsymbol{z} \rangle}(\boldsymbol{x}\boldsymbol{y}) \succeq f''_{\langle \boldsymbol{z} \rangle}(s)\boldsymbol{c} \tag{3.63}$$

and

$$f''_{\langle \boldsymbol{z} \rangle}(x_1) \succ f''_{\langle \boldsymbol{z} \rangle}(s). \tag{3.64}$$

By Lemma 2.3.1, we may assume

$$|f''^*_{\langle \boldsymbol{z}\boldsymbol{x} \rangle}(\boldsymbol{y})| \geq \max\{k, 1\}. \tag{3.65}$$

By (3.64) and Lemma 3.3.2 (ii), we obtain

$$f'_{\langle \boldsymbol{z} \rangle}(x_1) \succ f'_{\langle \boldsymbol{z} \rangle}(s). \tag{3.66}$$

This shows that $f'_{\langle \boldsymbol{z} \rangle}$ is not prefix-free, which conflicts with $F' \in \mathscr{F}_{k\text{-dec}}$ in the case $k = 0$ by Lemma 2.2.7. Thus, we consider the case $k \geq 1$, that is,

$$\boldsymbol{c} \neq \lambda. \tag{3.67}$$

Equation (3.63) leads to

$$f''^*_{\langle \boldsymbol{z} \rangle}(\boldsymbol{x}\boldsymbol{y}) \succeq f''_{\langle \boldsymbol{z} \rangle}(s)\boldsymbol{c} \overset{\text{(A)}}{\Longrightarrow} \psi_{\boldsymbol{z}}(f''^*_{\langle \boldsymbol{z} \rangle}(\boldsymbol{x}\boldsymbol{y})) \succeq \psi_{\boldsymbol{z}}(f''_{\langle \boldsymbol{z} \rangle}(s)\boldsymbol{c}) \tag{3.68}$$

$$\overset{\text{(B)}}{\Longleftrightarrow} \psi_{\boldsymbol{z}}(f''^*_{\langle \boldsymbol{z} \rangle}(\boldsymbol{x})f''^*_{\langle \boldsymbol{z}\boldsymbol{x} \rangle}(\boldsymbol{y})) \succeq \psi_{\boldsymbol{z}}(f''_{\langle \boldsymbol{z} \rangle}(s)\boldsymbol{c}) \tag{3.69}$$

$$\overset{\text{(C)}}{\Longleftrightarrow} f'^*_{\langle \boldsymbol{z} \rangle}(\boldsymbol{x})\psi_{\boldsymbol{z}\boldsymbol{x}}(f''^*_{\langle \boldsymbol{z}\boldsymbol{x} \rangle}(\boldsymbol{y})) \succeq f'_{\langle \boldsymbol{z} \rangle}(s)\psi_{\boldsymbol{z}s}(\boldsymbol{c}) \tag{3.70}$$

$$\overset{\text{(D)}}{\Longleftrightarrow} f'^*_{\langle \boldsymbol{z} \rangle}(\boldsymbol{x})f''^*_{\langle \boldsymbol{z}\boldsymbol{x} \rangle}(\boldsymbol{y}) \succeq f'_{\langle \boldsymbol{z} \rangle}(s)\psi_{\boldsymbol{z}s}(\boldsymbol{c}), \tag{3.71}$$

where (A) follows from Lemma 3.3.4 (i), (B) follows from Lemma 2.1.1 (i) and Lemma 3.3.1 (i), (C) follows from (3.65), (3.67), and the second case of (3.59), and (D) follows from Lemma 3.3.4 (iii) and $|\boldsymbol{zx}| = L$.

Then by (3.66) and (3.71), we have

$$f'^*_{\langle \boldsymbol{z} \rangle}(\boldsymbol{x})[f''^*_{\langle \boldsymbol{zx} \rangle}(\boldsymbol{y})]_k \succeq f'_{\langle \boldsymbol{z} \rangle}(s)[\psi_{\boldsymbol{z}s}(\boldsymbol{c})]_k. \tag{3.72}$$

Also, we have

$$[f''^*_{\langle \boldsymbol{zx} \rangle}(\boldsymbol{y})]_k \in \mathcal{P}^k_{F'',\langle \boldsymbol{zx} \rangle} \overset{\text{(A)}}{\subseteq} \mathcal{P}^k_{F',\langle \boldsymbol{zx} \rangle}, \tag{3.73}$$

where (A) follows from Lemma 3.3.3 (iii) and $\langle \boldsymbol{zx} \rangle \in \mathcal{S}^L \subseteq \mathcal{J}$. Hence, by (2.16) there exists $\boldsymbol{y}' \in \mathcal{S}^*$ such that $f'^*_{\langle \boldsymbol{zx} \rangle}(\boldsymbol{y}') \succeq [f''^*_{\langle \boldsymbol{zx} \rangle}(\boldsymbol{y})]_k$, which leads to

$$f'^*_{\langle \boldsymbol{z} \rangle}(\boldsymbol{xy}') = f'^*_{\langle \boldsymbol{z} \rangle}(\boldsymbol{x}) f'^*_{\langle \boldsymbol{zx} \rangle}(\boldsymbol{y}') \succeq f'^*_{\langle \boldsymbol{z} \rangle}(\boldsymbol{x})[f''^*_{\langle \boldsymbol{zx} \rangle}(\boldsymbol{y})]_k \overset{\text{(A)}}{\succeq} f'_{\langle \boldsymbol{z} \rangle}(s)[\psi_{\boldsymbol{z}s}(\boldsymbol{c})]_k, \tag{3.74}$$

where (A) follows from (3.72). Equations (3.66) and (3.74) show

$$[\psi_{\boldsymbol{z}s}(\boldsymbol{c})]_k \in \bar{\mathcal{P}}^k_{F',\langle \boldsymbol{z} \rangle}(f'_{\langle \boldsymbol{z} \rangle}(s)) \tag{3.75}$$

by (2.13).

On the other hand, by $\boldsymbol{c} \in \mathcal{P}^k_{F'',\langle \boldsymbol{z}s \rangle}$ and (2.16), there exist $\boldsymbol{x} \in \mathcal{S}^{L-|\boldsymbol{z}s|}$ and $\boldsymbol{y} \in \mathcal{S}^*$ such that

$$f''^*_{\langle \boldsymbol{z}s \rangle}(\boldsymbol{xy}) \succeq \boldsymbol{c}. \tag{3.76}$$

By Lemma 2.3.1, we may assume

$$|f''^*_{\langle \boldsymbol{z}s\boldsymbol{x} \rangle}(\boldsymbol{y})| \geq k \geq 1. \tag{3.77}$$

We have

$$f'^*_{\langle \boldsymbol{z}s \rangle}(\boldsymbol{x}) f''^*_{\langle \boldsymbol{z}s\boldsymbol{x} \rangle}(\boldsymbol{y}) \overset{\text{(A)}}{=} f'^*_{\langle \boldsymbol{z}s \rangle}(\boldsymbol{x}) \psi_{\boldsymbol{z}s\boldsymbol{x}}(f''^*_{\langle \boldsymbol{z}s\boldsymbol{x} \rangle}(\boldsymbol{y})) \overset{\text{(B)}}{=} \psi_{\boldsymbol{z}s}(f''^*_{\langle \boldsymbol{z}s \rangle}(\boldsymbol{xy})) \overset{\text{(C)}}{\succeq} \psi_{\boldsymbol{z}s}(\boldsymbol{c}), \tag{3.78}$$

where (A) follows from Lemma 3.3.4 (iii) and $|\boldsymbol{z}s\boldsymbol{x}| = L$, (B) follows from (3.77) and the second case of (3.59), and (C) follows from (3.76) and Lemma 3.3.4 (i).

Hence, we have

$$f'^{*}_{\langle \boldsymbol{z}s\rangle}(\boldsymbol{x})[f''^{*}_{\langle \boldsymbol{z}s\boldsymbol{x}\rangle}(\boldsymbol{y})]_k \succeq [\psi_{\boldsymbol{z}s}(\boldsymbol{c})]_k. \tag{3.79}$$

Also, we have

$$[f''^{*}_{\langle \boldsymbol{z}s\boldsymbol{x}\rangle}(\boldsymbol{y})]_k \in \mathcal{P}^k_{F'',\langle \boldsymbol{z}s\boldsymbol{x}\rangle} \overset{\text{(A)}}{\subseteq} \mathcal{P}^k_{F',\langle \boldsymbol{z}s\boldsymbol{x}\rangle}, \tag{3.80}$$

where (A) follows from Lemma 3.3.3 (iii) and $\langle \boldsymbol{z}s\boldsymbol{x}\rangle \in \mathcal{S}^L \subseteq \mathcal{J}$. Hence, there exists $\boldsymbol{y}' \in \mathcal{S}^*$ such that $f'^{*}_{\langle \boldsymbol{z}\boldsymbol{x}\rangle}(\boldsymbol{y}') \succeq [f''^{*}_{\langle \boldsymbol{z}s\boldsymbol{x}\rangle}(\boldsymbol{y})]_k$, which leads to

$$f'^{*}_{\langle \boldsymbol{z}s\rangle}(\boldsymbol{x}\boldsymbol{y}') = f'^{*}_{\langle \boldsymbol{z}s\rangle}(\boldsymbol{x})f'^{*}_{\langle \boldsymbol{z}\boldsymbol{x}s\rangle}(\boldsymbol{y}') \succeq f'^{*}_{\langle \boldsymbol{z}s\rangle}(\boldsymbol{x})[f''^{*}_{\langle \boldsymbol{z}s\boldsymbol{x}\rangle}(\boldsymbol{y})]_k \overset{\text{(A)}}{\succeq} [\psi_{\boldsymbol{z}s}(\boldsymbol{c})]_k, \tag{3.81}$$

where (A) follows from (3.79). This shows

$$[\psi_{\boldsymbol{z}s}(\boldsymbol{c})]_k \in \mathcal{P}^k_{F',\langle \boldsymbol{z}s\rangle} \tag{3.82}$$

by (2.16). By (3.75) and (3.82), the code-tuple F' does not satisfy Definition 2.2.3 (a), which conflicts with $F' \in \mathscr{F}_{k\text{-dec}}$.

Consequently, F'' satisfies Definition 2.2.3 (a).

Next, we show that F'' satisfies Definition 2.2.3 (b). Namely, we show that for any $i \in [F'']$ and $s, s' \in \mathcal{S}$ such that $s \neq s'$ and $f''_i(s) = f''_i(s')$, we have $\mathcal{P}^k_{F',\tau'_i(s)} \cap \mathcal{P}^k_{F',\tau'_i(s')} = \emptyset$. We prove for the following two cases: the case $i \in \mathcal{J}$ and the case $i \in [F''] \setminus \mathcal{J}$.

- The case $i \in \mathcal{J}$: Then for any $i \in \mathcal{J}$ and $s, s' \in \mathcal{S}$ such that $s \neq s'$ and $f''_i(s) = f''_i(s')$, we have

$$f'_i(s) = f'_i(s') \tag{3.83}$$

by Lemma 3.3.3 (ii), and we have

$$\mathcal{P}^k_{F'',\tau''_i(s)} \cap \mathcal{P}^k_{F'',\tau''_i(s')} \overset{\text{(A)}}{\subseteq} \mathcal{P}^k_{F',\tau''_i(s)} \cap \mathcal{P}^k_{F',\tau''_i(s')} \overset{\text{(B)}}{=} \mathcal{P}^k_{F',\tau'_i(s)} \cap \mathcal{P}^k_{F',\tau'_i(s')} \overset{\text{(C)}}{=} \emptyset, \tag{3.84}$$

where (A) follows from Lemma 3.3.3 (i) (iii) since $\tau''_i(s), \tau''_i(s') \in [F]$, (B) follows from (3.36), and (C) follows from $F' \in \mathscr{F}_{k\text{-dec}}$ and (3.83).

- The case $i \in [F''] \setminus \mathcal{J}$: We prove by contradiction assuming that there exists $\boldsymbol{z} \in \mathcal{S}^{\leq L-1}$, $s, s' \in \mathcal{S}$, and $\boldsymbol{c} \in \mathcal{P}^k_{F'',\langle \boldsymbol{z}s\rangle} \cap \mathcal{P}^k_{F'',\langle \boldsymbol{z}s'\rangle}$ such that $s \neq s'$ and

$$f''_{\langle \boldsymbol{z}\rangle}(s) = f''_{\langle \boldsymbol{z}\rangle}(s'). \tag{3.85}$$

By the similar way to derive (3.82), we obtain

$$[\psi_{\boldsymbol{z}s}(\boldsymbol{c})]_k \in \mathcal{P}^k_{F',\langle \boldsymbol{z}s\rangle} \tag{3.86}$$

from $\boldsymbol{c} \in \mathcal{P}^k_{F'',\langle \boldsymbol{z}s\rangle}$. By (3.85) and Lemma 3.3.4 (i), we have

$$\psi_{\langle \boldsymbol{z}\rangle}(f''_{\langle \boldsymbol{z}\rangle}(s)) = \psi_{\langle \boldsymbol{z}\rangle}(f''_{\langle \boldsymbol{z}\rangle}(s')). \tag{3.87}$$

By Lemma 3.3.4 (ii), exactly one of $f'_{\langle \boldsymbol{z}\rangle}(s) = f'_{\langle \boldsymbol{z}\rangle}(s')$, $f'_{\langle \boldsymbol{z}\rangle}(s) \prec f'_{\langle \boldsymbol{z}\rangle}(s')$, and $f'_{\langle \boldsymbol{z}\rangle}(s) \succ f'_{\langle \boldsymbol{z}\rangle}(s')$ holds. Therefore, $f'_{\langle \boldsymbol{z}\rangle}$ is not prefix-free, which conflicts with $F' \in \mathscr{F}_{k\text{-dec}}$ in the case $k = 0$ by Lemma 2.2.7. We consider the case $k \geq 1$, that is,

$$\boldsymbol{c} \neq \lambda. \tag{3.88}$$

We consider the following two cases separately: the case $f'_{\langle \boldsymbol{z}\rangle}(s) = f'_{\langle \boldsymbol{z}\rangle}(s')$ and the case $f'_{\langle \boldsymbol{z}\rangle}(s) \prec f'_{\langle \boldsymbol{z}\rangle}(s')$. Note that we may exclude the case $f'_{\langle \boldsymbol{z}\rangle}(s) \succ f'_{\langle \boldsymbol{z}\rangle}(s')$ by symmetry.

- The case $f'_{\langle \boldsymbol{z}\rangle}(s) = f'_{\langle \boldsymbol{z}\rangle}(s')$: By (3.58), we have $\psi_{\boldsymbol{z}s}(\boldsymbol{c}) = \psi_{\boldsymbol{z}s'}(\boldsymbol{c})$ and thus

$$[\psi_{\boldsymbol{z}s}(\boldsymbol{c})]_k = [\psi_{\boldsymbol{z}s'}(\boldsymbol{c})]_k \overset{\text{(A)}}{\in} \mathcal{P}^k_{F',\langle \boldsymbol{z}s'\rangle}, \tag{3.89}$$

  where (A) is obtained from $\boldsymbol{c} \in \mathcal{P}^k_{F'',\langle \boldsymbol{z}s'\rangle}$ by the similar way to derive (3.82).

  By (3.86), (3.89), and $f'_{\langle \boldsymbol{z}\rangle}(s) = f'_{\langle \boldsymbol{z}\rangle}(s')$, the code-tuple F' does not satisfy Definition 2.2.3 (b), which conflicts with $F' \in \mathscr{F}_{k\text{-dec}}$.
- The case $f'_{\langle \boldsymbol{z}\rangle}(s) \prec f'_{\langle \boldsymbol{z}\rangle}(s')$: Then by (3.87) and Lemma 3.3.4 (ii), it must hold that

$$f'^{*}_{\langle \lambda\rangle}(\boldsymbol{z}) \prec f'^{*}_{\langle \lambda\rangle}(\boldsymbol{z}s') = \boldsymbol{d} \tag{3.90}$$

  and

$$f'_{\langle \boldsymbol{z}\rangle}(s) = \mathrm{pref}(f'_{\langle \boldsymbol{z}\rangle}(s')). \tag{3.91}$$

Thus, we have

$$
\begin{aligned}
f'_{\langle \boldsymbol{z} \rangle}(s)d_l &\overset{\text{(A)}}{=} \mathrm{pref}(f'_{\langle \boldsymbol{z} \rangle}(s'))d_l && (3.92)\\
&= f'^{*}_{\langle \lambda \rangle}(\boldsymbol{z})^{-1} f'^{*}_{\langle \lambda \rangle}(\boldsymbol{z})\mathrm{pref}(f'_{\langle \boldsymbol{z} \rangle}(s'))d_l && (3.93)\\
&\overset{\text{(B)}}{=} f'^{*}_{\langle \lambda \rangle}(\boldsymbol{z})^{-1}\mathrm{pref}(f'^{*}_{\langle \lambda \rangle}(\boldsymbol{z}s'))d_l && (3.94)\\
&\overset{\text{(C)}}{=} f'^{*}_{\langle \lambda \rangle}(\boldsymbol{z})^{-1}\mathrm{pref}(\boldsymbol{d})d_l && (3.95)\\
&= f'^{*}_{\langle \lambda \rangle}(\boldsymbol{z})^{-1}\boldsymbol{d} && (3.96)\\
&\overset{\text{(D)}}{=} f'^{*}_{\langle \lambda \rangle}(\boldsymbol{z})^{-1} f'^{*}_{\langle \lambda \rangle}(\boldsymbol{z}s') && (3.97)\\
&\overset{\text{(E)}}{=} f'^{*}_{\langle \lambda \rangle}(\boldsymbol{z})^{-1} f'^{*}_{\langle \lambda \rangle}(\boldsymbol{z}) f'_{\langle \boldsymbol{z} \rangle}(s') && (3.98)\\
&= f'_{\langle \boldsymbol{z} \rangle}(s'), && (3.99)
\end{aligned}
$$

where (A) follows from (3.91), (B) follows from Lemma 2.1.1 (i) and Lemma 3.3.1 (i), (C) follows from (3.90), (D) follows from (3.90), and (E) follows from Lemma 2.1.1 (i) and Lemma 3.3.1 (i).

Also, we have

$$
\begin{aligned}
\mathrm{pref}(\boldsymbol{d}) &\overset{\text{(A)}}{=} \mathrm{pref}(f'^{*}_{\langle \lambda \rangle}(\boldsymbol{z}s')) && (3.100)\\
&= \mathrm{pref}(f'^{*}_{\langle \lambda \rangle}(\boldsymbol{z}) f'_{\langle \boldsymbol{z} \rangle}(s')) && (3.101)\\
&\overset{\text{(B)}}{=} \mathrm{pref}(f'^{*}_{\langle \lambda \rangle}(\boldsymbol{z}) f'_{\langle \boldsymbol{z} \rangle}(s)d_l) && (3.102)\\
&= f'^{*}_{\langle \lambda \rangle}(\boldsymbol{z}) f'_{\langle \boldsymbol{z} \rangle}(s) && (3.103)\\
&= f'^{*}_{\langle \lambda \rangle}(\boldsymbol{z}s), && (3.104)
\end{aligned}
$$

where (A) follows from (3.90), and (B) follows from (3.99).

By $\boldsymbol{c} \in \mathcal{P}^{k}_{F'',\langle \boldsymbol{z}s' \rangle}$ and (2.16), there exist $\boldsymbol{x} \in \mathcal{S}^{L-|\boldsymbol{z}s'|}$ and $\boldsymbol{y} \in \mathcal{S}^{*}$ such that

$$f''^{*}_{\langle \boldsymbol{z}s' \rangle}(\boldsymbol{x}\boldsymbol{y}) \succeq \boldsymbol{c}. \tag{3.105}$$

By Lemma 2.3.1, we may assume

$$|f''^{*}_{\langle \boldsymbol{z}s'\boldsymbol{x} \rangle}(\boldsymbol{y})| \geq k \geq 1. \tag{3.106}$$

We have

$$
\begin{aligned}
&f'_{\langle \boldsymbol{z}\rangle}(s')f'^{*}_{\langle \boldsymbol{z}s'\rangle}(\boldsymbol{x})f''^{*}_{\langle \boldsymbol{z}s'\boldsymbol{x}\rangle}(\boldsymbol{y}) \\
&\overset{\text{(A)}}{=} f'_{\langle \boldsymbol{z}\rangle}(s')\psi_{\boldsymbol{z}s'}(f''^{*}_{\langle \boldsymbol{z}s'\rangle}(\boldsymbol{x})f''^{*}_{\langle \boldsymbol{z}s'\boldsymbol{x}\rangle}(\boldsymbol{y})) && (3.107)\\
&\overset{\text{(B)}}{=} f'_{\langle \boldsymbol{z}\rangle}(s')\psi_{\boldsymbol{z}s'}(f''^{*}_{\langle \boldsymbol{z}s'\rangle}(\boldsymbol{x}\boldsymbol{y})) && (3.108)\\
&\overset{\text{(C)}}{\succeq} f'_{\langle \boldsymbol{z}\rangle}(s')\psi_{\boldsymbol{z}s'}(\boldsymbol{c}) && (3.109)\\
&\overset{\text{(D)}}{=} f'_{\langle \boldsymbol{z}\rangle}(s)d_l\psi_{\boldsymbol{z}s'}(\boldsymbol{c}) && (3.110)\\
&\overset{\text{(E)}}{=} f'_{\langle \boldsymbol{z}\rangle}(s)d_l\boldsymbol{c} && (3.111)\\
&= f'_{\langle \boldsymbol{z}\rangle}(s)\mathrm{pref}(\boldsymbol{d})^{-1}\boldsymbol{d}\mathrm{pref}(\boldsymbol{d})^{-1}(\mathrm{pref}(\boldsymbol{d})\boldsymbol{c}) && (3.112)\\
&\overset{\text{(F)}}{=} f'_{\langle \boldsymbol{z}\rangle}(s)f'^{*}_{\langle \lambda\rangle}(\boldsymbol{z}s)^{-1}\boldsymbol{d}\mathrm{pref}(\boldsymbol{d})^{-1}(f'^{*}_{\langle \lambda\rangle}(\boldsymbol{z}s)\boldsymbol{c}) && (3.113)\\
&\overset{\text{(G)}}{=} f'_{\langle \boldsymbol{z}\rangle}(s)\psi_{\boldsymbol{z}s}(\boldsymbol{c}), && (3.114)
\end{aligned}
$$

where (A) follows from (3.106) and the second case of (3.59), (B) follows from Lemma 2.1.1 (i) and Lemma 3.3.1 (i), (C) follows from (3.105) and Lemma 3.3.4 (i), (D) follows from (3.99), (E) follows from the second case of (3.58) because $f'^{*}_{\langle \lambda\rangle}(\boldsymbol{z}s') \preceq \mathrm{pref}(\boldsymbol{d})$ does not hold by (3.90), (F) follows from (3.104), and (G) follows from the first case of (3.58) because $f'^{*}_{\langle \lambda\rangle}(\boldsymbol{z}s) = \mathrm{pref}(f'^{*}_{\langle \lambda\rangle}(\boldsymbol{z}s')) = \mathrm{pref}(\boldsymbol{d}) \prec f'^{*}_{\langle \lambda\rangle}(\boldsymbol{z}s)\boldsymbol{c}$ by (3.90), (3.91), and (3.88).
Hence, by $f'_{\langle \boldsymbol{z}\rangle}(s) \prec f'_{\langle \boldsymbol{z}\rangle}(s')$, we have

$$
f'_{\langle \boldsymbol{z}\rangle}(s')f'^{*}_{\langle \boldsymbol{z}s'\rangle}(\boldsymbol{x})[f''^{*}_{\langle \boldsymbol{z}s'\boldsymbol{x}\rangle}(\boldsymbol{y})]_k \succeq f'_{\langle \boldsymbol{z}\rangle}(s)[\psi_{\boldsymbol{z}s}(\boldsymbol{c})]_k. \tag{3.115}
$$

Also, we have

$$
[f''^{*}_{\langle \boldsymbol{z}s'\boldsymbol{x}\rangle}(\boldsymbol{y})]_k \in \mathcal{P}^k_{F'',\langle \boldsymbol{z}s'\boldsymbol{x}\rangle} \overset{\text{(A)}}{\subseteq} \mathcal{P}^k_{F',\langle \boldsymbol{z}s'\boldsymbol{x}\rangle}, \tag{3.116}
$$

where (A) follows from Lemma 3.3.3 (iii) and $\langle \boldsymbol{z}s'\boldsymbol{x}\rangle \in \mathcal{S}^L \subseteq \mathcal{J}$. Hence, there exists $\boldsymbol{y}' \in \mathcal{S}^*$ such that $f'^{*}_{\langle \boldsymbol{z}s'\boldsymbol{x}\rangle}(\boldsymbol{y}') \succeq [f''^{*}_{\langle \boldsymbol{z}s'\boldsymbol{x}\rangle}(\boldsymbol{y})]_k$, which leads to

$$
\begin{aligned}
f'^{*}_{\langle \boldsymbol{z}\rangle}(s'\boldsymbol{x}\boldsymbol{y}') &= f'_{\langle \boldsymbol{z}\rangle}(s')f'^{*}_{\langle \boldsymbol{z}s'\rangle}(\boldsymbol{x})f'^{*}_{\langle \boldsymbol{z}s'\boldsymbol{x}\rangle}(\boldsymbol{y}') && (3.117)\\
&\succeq f'_{\langle \boldsymbol{z}\rangle}(s')f'^{*}_{\langle \boldsymbol{z}s'\rangle}(\boldsymbol{x})[f'^{*}_{\langle \boldsymbol{z}s'\boldsymbol{x}\rangle}(\boldsymbol{y}')]_k && (3.118)\\
&\overset{\text{(A)}}{\succeq} f'_{\langle \boldsymbol{z}\rangle}(s)[\psi_{\boldsymbol{z}s}(\boldsymbol{c})]_k, && (3.119)
\end{aligned}
$$

where (A) follows from (3.115). The assumption that $f'_{\langle \boldsymbol{z} \rangle}(s) \prec f'_{\langle \boldsymbol{z} \rangle}(s')$ and (3.117) shows that

$$[\psi_{\boldsymbol{z}s}(\boldsymbol{c})]_k \in \bar{\mathcal{P}}^k_{F',\langle \boldsymbol{z} \rangle}(f'_{\langle \boldsymbol{z} \rangle}(s)) \tag{3.120}$$

by (2.13). By (3.86) and (3.120), the code-tuple F' does not satisfy Definition 2.2.3 (a), which conflicts with $F' \in \mathscr{F}_{k\text{-dec}}$.

Consequently, F'' satisfies Definition 2.2.3 (b). □

3.4 Proof of Theorem 3.1.3

The outline of the proof is as follows. First, we define an operation called *rotation* which transforms a code-tuple $F \in \mathscr{F}_{\mathrm{ext}}$ into another code-tuple $\widehat{F} \in \mathscr{F}$. Next, we show that for any $F \in \mathscr{F}_{\mathrm{reg}} \cap \mathscr{F}_{\mathrm{ext}} \cap \mathscr{F}_{k\text{-dec}}$, we have $\widehat{F} \in \mathscr{F}_{\mathrm{reg}} \cap \mathscr{F}_{\mathrm{ext}} \cap \mathscr{F}_{k\text{-dec}}$ and $L(\widehat{F}) = L(F)$. Namely, the rotation preserves "the key properties" of a code-tuple. Then we prove Theorem 3.1.3 by showing that any $F \in \mathscr{F}_{\mathrm{reg}} \cap \mathscr{F}_{\mathrm{ext}} \cap \mathscr{F}_{k\text{-dec}}$ can be transformed into some $F' \in \mathscr{F}_{\mathrm{reg}} \cap \mathscr{F}_{\mathrm{ext}} \cap \mathscr{F}_{k\text{-dec}} \cap \mathscr{F}_{\mathrm{fork}}$ by rotation in a repetitive manner without changing the average codeword length.

Definition 3.4.1. *For $F(f, \tau) \in \mathscr{F}_{\mathrm{ext}}$, we define $\widehat{F}(\widehat{f}, \widehat{\tau}) \in \mathscr{F}$ as follows.*
For $i \in [F]$ and $s \in \mathcal{S}$,

$$\widehat{f}_i(s) = \begin{cases} f_i(s) d_{F,\tau_i(s)} & \text{if } \mathcal{P}^1_{F,i} = \{0,1\}, \\ \mathrm{suff}(f_i(s) d_{F,\tau_i(s)}) & \text{if } \mathcal{P}^1_{F,i} \neq \{0,1\}, \end{cases} \tag{3.121}$$

$$\widehat{\tau}_i(s) = \tau_i(s), \tag{3.122}$$

where

$$d_{F,i} = \begin{cases} 0 & \text{if } \mathcal{P}^1_{F,i} = \{0\}, \\ 1 & \text{if } \mathcal{P}^1_{F,i} = \{1\}, \\ \lambda & \text{if } \mathcal{P}^1_{F,i} = \{0,1\}. \end{cases} \tag{3.123}$$

The operation which transforms a given $F \in \mathscr{F}_{\mathrm{ext}}$ into $\widehat{F} \in \mathscr{F}$ defined above is called rotation.

Example 3.4.1. *Table 3.4 shows $\mathcal{P}^1_{F,i}, i \in [F]$ and $d_{F,i}, i \in [F]$ for the code-tuples $F^{(\gamma)}, F^{(\delta)}$, and $F^{(\epsilon)}$ in Table 3.3.*

Table 3.3: Examples of a code-tuple $F^{(\gamma)}, F^{(\delta)}$, and $F^{(\epsilon)}$

$s \in \mathcal{S}$	$f_0^{(\gamma)}$	$\tau_0^{(\gamma)}$	$f_1^{(\gamma)}$	$\tau_1^{(\gamma)}$	$f_2^{(\gamma)}$	$\tau_2^{(\gamma)}$
a	01	0	00	1	1100	1
b	10	1	λ	0	1110	0
c	0100	0	00111	1	111000	2
d	01	2	00111	2	110	2

$s \in \mathcal{S}$	$f_0^{(\delta)}$	$\tau_0^{(\delta)}$	$f_1^{(\delta)}$	$\tau_1^{(\delta)}$	$f_2^{(\delta)}$	$\tau_2^{(\delta)}$
a	01	0	00	1	100	1
b	10	1	λ	0	110	0
c	0100	0	00111	1	110001	2
d	011	2	001111	2	101	2

$s \in \mathcal{S}$	$f_0^{(\epsilon)}$	$\tau_0^{(\epsilon)}$	$f_1^{(\epsilon)}$	$\tau_1^{(\epsilon)}$	$f_2^{(\epsilon)}$	$\tau_2^{(\epsilon)}$
a	01	0	00	1	00	1
b	10	1	λ	0	10	0
c	0100	0	00111	1	100011	2
d	0111	2	0011111	2	011	2

Example 3.4.2. *In Table 3.3, $F^{(\delta)}$ is obtained by applying rotation to $F^{(\gamma)}$, that is, $F^{(\delta)} = \widehat{F^{(\gamma)}}$. Also, $F^{(\epsilon)}$ is obtained by applying rotation to $F^{(\delta)}$, that is, $F^{(\epsilon)} = \widehat{F^{(\delta)}}$. Furthermore, we obtain $F^{(\epsilon)}$ itself by applying rotation to $F^{(\epsilon)}$, that is, $F^{(\epsilon)} = \widehat{F^{(\epsilon)}}$.*

Directly from Definition 3.4.1, we can see that for any $F(f, \tau) \in \mathscr{F}_{\mathrm{ext}}, i \in [F]$, and $s \in \mathcal{S}$, we have

$$d_{F,i}\widehat{f}_i(s) = f_i(s)d_{F,\tau_i(s)}. \tag{3.124}$$

This relation (3.124) is generalized to the following Lemma 3.4.1.

Table 3.4: $\mathcal{P}^1_{F,i}, i \in [F]$ and $d_{F,i}, i \in [F]$ for the code-tuples $F^{(\gamma)}, F^{(\delta)}$, and $F^{(\epsilon)}$ in Table 3.3

F	$\mathcal{P}^1_{F,0}$	$d_{F,0}$	$\mathcal{P}^1_{F,1}$	$d_{F,1}$	$\mathcal{P}^1_{F,2}$	$d_{F,2}$
$F^{(\gamma)}$	$\{0,1\}$	λ	$\{0,1\}$	λ	$\{1\}$	1
$F^{(\delta)}$	$\{0,1\}$	λ	$\{0,1\}$	λ	$\{1\}$	1
$F^{(\epsilon)}$	$\{0,1\}$	λ	$\{0,1\}$	λ	$\{0,1\}$	λ

Lemma 3.4.1. *For any $F(f,\tau) \in \mathscr{F}_{\text{ext}}, i \in [F]$, and $\boldsymbol{x} \in \mathcal{S}^*$, we have*

$$d_{F,i}\widehat{f}_i^*(\boldsymbol{x}) = f_i^*(\boldsymbol{x})d_{F,\tau_i^*(\boldsymbol{x})}. \tag{3.125}$$

Proof of Lemma 3.4.1. We prove the lemma by induction for $|\boldsymbol{x}|$.

For the base case $|\boldsymbol{x}| = 0$, we have

$$d_{F,i}\widehat{f}_i^*(\boldsymbol{x}) = d_{F,i}\widehat{f}_i^*(\lambda) \overset{\text{(A)}}{=} d_{F,i}\lambda = \lambda d_{F,i} \overset{\text{(B)}}{=} f_i^*(\lambda)d_{F,\tau_i^*(\lambda)} = f_i^*(\boldsymbol{x})d_{F,\tau_i^*(\boldsymbol{x})} \tag{3.126}$$

as desired, where (A) and (B) follow from (2.4).

Let $l \geq 1$ and assume that (3.125) holds for any $\boldsymbol{x}' \in \mathcal{S}^*$ such that $|\boldsymbol{x}'| < l$ as the induction hypothesis. We prove that (3.125) holds for any $\boldsymbol{x} \in \mathcal{S}^l$. We have

$$\begin{aligned} d_{F,i}\widehat{f}_i^*(\boldsymbol{x}) &\overset{\text{(A)}}{=} d_{F,i}\widehat{f}_i(x_1)\widehat{f^*}_{\tau_i(x_1)}(\text{suff}(\boldsymbol{x})) & (3.127)\\ &\overset{\text{(B)}}{=} f_i(x_1)d_{F,\tau_i(x_1)}\widehat{f^*}_{\tau_i(x_1)}(\text{suff}(\boldsymbol{x})) & (3.128)\\ &\overset{\text{(C)}}{=} f_i(x_1)f^*_{\tau_i(x_1)}(\text{suff}(\boldsymbol{x}))d_{F,\tau^*_{\tau_i(x_1)}(\text{suff}(\boldsymbol{x}))} & (3.129)\\ &\overset{\text{(D)}}{=} f_i(x_1)f^*_{\tau_i(x_1)}(\text{suff}(\boldsymbol{x}))d_{F,\tau_i^*(\boldsymbol{x})} & (3.130)\\ &\overset{\text{(E)}}{=} f_i^*(\boldsymbol{x})d_{F,\tau_i^*(\boldsymbol{x})} & (3.131) \end{aligned}$$

as desired, where (A) follows from (2.4), (B) follows from (3.124), (C) follows from the induction hypothesis, (D) follows from (2.5), and (E) follows from from (2.4). □

Example 3.4.3. *For $F(f,\tau) \coloneqq F^{(\gamma)}$ of Table 3.3, we have $\widehat{F} = F^{(\delta)}$ as seen in Example 3.4.2. We can see $d_{F,2}\widehat{f}_2^*(\text{bbc}) = 1\widehat{f}_2(\text{b})\widehat{f}_0(\text{b})\widehat{f}_1(\text{c}) = 11101000$ and $f_2^*(\text{bbc})d_{F,\tau_2^*(\text{bbc})} = f_2^*(\text{bbc})d_{F,1} = f_2(\text{b})f_0(\text{b})f_1(\text{c}) = 11101000$. Hence, we confirm $d_{F,2}\widehat{f}_2^*(\text{bbc}) = f_2^*(\text{bbc})d_{F,\tau_2^*(\text{bbc})}$.*

Next, we prove that if $F \in \mathscr{F}_{\text{reg}} \cap \mathscr{F}_{\text{ext}} \cap \mathscr{F}_{k\text{-dec}}$, then $\widehat{F} \in \mathscr{F}_{\text{reg}} \cap \mathscr{F}_{\text{ext}} \cap \mathscr{F}_{k\text{-dec}}$ and $L(\widehat{F}) = L(F)$. To prove it, we show the following Lemmas 3.4.2–3.4.4.

Lemma 3.4.2. *For any integer $k \geq 0$ and $F(f,\tau) \in \mathscr{F}_{k\text{-dec}} \cap \mathscr{F}_{\text{ext}}$, we have $\widehat{F}(\widehat{f},\widehat{\tau}) \in \mathscr{F}_{k\text{-dec}}$.*

Lemma 3.4.3. *For any $F(f,\tau) \in \mathscr{F}_{\text{ext}}$, we have $\widehat{F}(\widehat{f},\widehat{\tau}) \in \mathscr{F}_{\text{ext}}$.*

Lemma 3.4.4. *For any $F \in \mathscr{F}_{\mathrm{reg}}$, we have $\widehat{F} \in \mathscr{F}_{\mathrm{reg}}$ and $L(\widehat{F}) = L(F)$.*

The proof of Lemma 3.4.2 relies on Lemma 3.4.5 whose proof is relegated to Subsection 3.5.9.

Lemma 3.4.5. *Let $F(f,\tau) \in \mathscr{F}_{\mathrm{ext}}$. There exists no tuple $(k, i, \boldsymbol{x}, \boldsymbol{x}')$ satisfying all of the following conditions (a)–(c), where k is a non-negative integer, $i \in [F]$, and $\boldsymbol{x}, \boldsymbol{x}' \in \mathcal{S}^*$.*

(a) $F \in \mathscr{F}_{k\text{-dec}}$.

(b) $|\widehat{f}_i^*(\boldsymbol{x})| + k \leq |\widehat{f}_i^*(\boldsymbol{x}')|$.

(c) $\boldsymbol{x}' \prec \boldsymbol{x}$.

Proof of Lemma 3.4.2. Fix $i \in [F]$ and $(\boldsymbol{x}, \boldsymbol{c}) \in \mathcal{S}^* \times \mathcal{C}^k$ arbitrarily. Also, choose $\boldsymbol{x}' \in \mathcal{S}^*$ such that

$$\widehat{f}_i^*(\boldsymbol{x})\boldsymbol{c} \preceq \widehat{f}_i^*(\boldsymbol{x}') \tag{3.132}$$

arbitrarily. Then, we have $d_{F,i}\widehat{f}_i^*(\boldsymbol{x})\boldsymbol{c} \preceq d_{F,i}\widehat{f}_i^*(\boldsymbol{x}')$. From Lemma 3.4.1, we have $f_i^*(\boldsymbol{x})d_{F,\tau_i^*(\boldsymbol{x})}\boldsymbol{c} \preceq f_i^*(\boldsymbol{x}')d_{F,\tau_i^*(\boldsymbol{x}')}$. By (3.123), there exists $\boldsymbol{x}'' \in \mathcal{S}^*$ such that

$$d_{F,\tau_i^*(\boldsymbol{x}')} \preceq f^*_{\tau_i^*(\boldsymbol{x}')}(\boldsymbol{x}''). \tag{3.133}$$

Defining $\boldsymbol{c}' \coloneqq [d_{F,\tau_i^*(\boldsymbol{x})}\boldsymbol{c}]_k$, we have

$$\begin{aligned}
f_i^*(\boldsymbol{x})\boldsymbol{c}' &\preceq f_i^*(\boldsymbol{x})d_{F,\tau_i^*(\boldsymbol{x})}\boldsymbol{c} &(3.134)\\
&\overset{\text{(A)}}{=} d_{F,i}\widehat{f}_i^*(\boldsymbol{x})\boldsymbol{c} &(3.135)\\
&\overset{\text{(B)}}{\preceq} d_{F,i}\widehat{f}_i^*(\boldsymbol{x}') &(3.136)\\
&\overset{\text{(C)}}{=} f_i^*(\boldsymbol{x}')d_{F,\tau_i^*(\boldsymbol{x}')} &(3.137)\\
&\overset{\text{(D)}}{\preceq} f_i^*(\boldsymbol{x}')f^*_{\tau_i^*(\boldsymbol{x}')}(\boldsymbol{x}'') &(3.138)\\
&\overset{\text{(E)}}{=} f_i^*(\boldsymbol{x}'\boldsymbol{x}''), &(3.139)
\end{aligned}$$

where (A) follows from Lemma 3.4.1, (B) follows from (3.132), (C) follows from Lemma 3.4.1, (D) follows from (3.133), and (E) follows Lemma 2.1.1 (i).

In general, exactly one of the following conditions holds: $\boldsymbol{x} \preceq \boldsymbol{x}'$; $\boldsymbol{x} \succ \boldsymbol{x}$; $\boldsymbol{x} \not\succeq \boldsymbol{x}'$. Now, $\boldsymbol{x} \succ \boldsymbol{x}'$ is impossible because if we assume $\boldsymbol{x} \succ \boldsymbol{x}'$, then the tuple $(k, i, \boldsymbol{x})$ satisfies the conditions (a)–(c) of Lemma 3.4.5, where the condition (b) follows from by (3.132). Thus, it suffices to consider the case where either $\boldsymbol{x} \preceq \boldsymbol{x}'$ or $\boldsymbol{x} \not\succeq \boldsymbol{x}'$ holds.

By $F \in \mathscr{F}_{k\text{-dec}}$, the pair $(\boldsymbol{x}, \boldsymbol{c}')$ is f_i^*-positive or f_i^*-negative. If $(\boldsymbol{x}, \boldsymbol{c}')$ is f_i^*-positive (resp. f_i^*-negative), then we have $\boldsymbol{x} \preceq \boldsymbol{x}'\boldsymbol{x}''$ (resp. $\boldsymbol{x} \not\preceq \boldsymbol{x}'\boldsymbol{x}''$) by (3.139). This implies that $\boldsymbol{x} \preceq \boldsymbol{x}'$ (resp. $\boldsymbol{x} \not\succeq \boldsymbol{x}'$) holds since $\boldsymbol{x} \succ \boldsymbol{x}'$ is impossible. Since $\boldsymbol{x}'$ is chosen arbitrarily, the pair $(\boldsymbol{x}, \boldsymbol{c})$ is $\widehat{f}_i^*$-positive (resp. $\widehat{f}_i^*$-negative), respectively. Therefore, we have $\widehat{F} \in \mathscr{F}_{k\text{-dec}}$. □

Proof of Lemma 3.4.3. Fix $i \in [F]$ arbitrarily. By $F \in \mathscr{F}_{\text{ext}}$ and Lemma 2.3.1, there exists $\boldsymbol{x} \in \mathcal{S}^*$ such that $|f_i^*(\boldsymbol{x})| \geq 2$. For such $\boldsymbol{x}$, we have

$$|\widehat{f}_i^*(\boldsymbol{x})| \overset{\text{(A)}}{=} |f_i^*(\boldsymbol{x})| + |d_{F,\tau_i^*(\boldsymbol{x})}| - |d_{F,i}| \overset{\text{(B)}}{\geq} 1, \tag{3.140}$$

where (A) follows since $d_{F,i}\widehat{f}_i^*(\boldsymbol{x}) = f_i^*(\boldsymbol{x})d_{F,\tau_i^*(\boldsymbol{x})}$ by Lemma 3.4.1, and (B) follows from $|f_i^*(\boldsymbol{x})| \geq 2, |d_{F,\tau_i^*(\boldsymbol{x})}| \geq 0$, and $|d_{F,i}| \leq 1$. Therefore, we have $\widehat{F} \in \mathscr{F}_{\text{ext}}$. □

Proof of Lemma 3.4.4. By (3.122), for any $i, j \in [F]$, we have $Q_{i,j}(\widehat{F}) = Q_{i,j}(F)$ (cf. Remark 2.4.1). Thus, we have $\widehat{F} \in \mathscr{F}_{\text{reg}}$, and for any $i \in [F]$, we have

$$\pi_i(\widehat{F}) = \pi_i(F). \tag{3.141}$$

Also, for any $i \in [F]$, we have

$$L_i(\widehat{F}) \overset{\text{(A)}}{=} \begin{cases} \sum_{s\in\mathcal{S}} |f_i(s)d_{F,\tau_i(s)}| \cdot \mu(s) & \text{if } i \notin \mathcal{B}, \\ \sum_{s\in\mathcal{S}} |\mathrm{suff}(f_i(s)d_{F,\tau_i(s)})| \cdot \mu(s) & \text{if } i \in \mathcal{B}, \end{cases} \tag{3.142}$$

$$= \begin{cases} L_i(F) + \sum_{s\in\mathcal{S}} |d_{F,\tau_i(s)}| \cdot \mu(s) & \text{if } i \notin \mathcal{B} \\ L_i(F) + \sum_{s\in\mathcal{S}} |d_{F,\tau_i(s)}| \cdot \mu(s) - 1 & \text{if } i \in \mathcal{B}, \end{cases} \tag{3.143}$$

$$\overset{\text{(B)}}{=} \begin{cases} L_i(F) + \sum_{j\in\mathcal{B}} Q_{i,j}(F) & \text{if } i \notin \mathcal{B}, \\ L_i(F) + \sum_{j\in\mathcal{B}} Q_{i,j}(F) - 1 & \text{if } i \in \mathcal{B}, \end{cases} \tag{3.144}$$

where $\mathcal{B} := \{i \in [F] : \mathcal{P}_{F,i}^1 \neq \{0, 1\}\}$, (A) follows from (3.123), and (B) follows from (3.121). Therefore, we obtain

$$L(\widehat{F}) = \sum_{i\in[F]} \pi_i(\widehat{F})L_i(\widehat{F}) \tag{3.145}$$

$$= \sum_{i\in[F]\setminus\mathcal{B}} \pi_i(\widehat{F})L_i(\widehat{F}) + \sum_{i\in\mathcal{B}} \pi_i(\widehat{F})L_i(\widehat{F}) \tag{3.146}$$

$$\overset{\text{(A)}}{=} \sum_{i\in[F]\setminus\mathcal{B}} \pi_i(F)(L_i(F) + \sum_{j\in\mathcal{B}} Q_{i,j}(F)) + \sum_{i\in\mathcal{B}} \pi_i(F)(L_i(F) + \sum_{j\in\mathcal{B}} Q_{i,j}(F) - 1) \tag{3.147}$$

$$= \sum_{i\in[F]\setminus\mathcal{B}} \pi_i(F)L_i(F) + \sum_{i\in\mathcal{B}} \pi_i(F)L_i(F) + \sum_{i\in[F]\setminus\mathcal{B}} \sum_{j\in\mathcal{B}} \pi_i(F)Q_{i,j}(F) + \sum_{i\in\mathcal{B}} \sum_{j\in\mathcal{B}} \pi_i(F)Q_{i,j}(F) - \sum_{i\in\mathcal{B}} \pi_i(F) \tag{3.148}$$

$$= \sum_{i\in[F]} \pi_i(F)L_i(F) + \sum_{i\in[F]} \sum_{j\in\mathcal{B}} \pi_i(F)Q_{i,j}(F) - \sum_{j\in\mathcal{B}} \pi_j(F) \tag{3.149}$$

$$\overset{\text{(B)}}{=} \sum_{i\in[F]} \pi_i(F)L_i(F) + \sum_{i\in[F]} \sum_{j\in\mathcal{B}} \pi_i(F)Q_{i,j}(F) - \sum_{j\in\mathcal{B}} \sum_{i\in[F]} \pi_i(F)Q_{i,j}(F) \tag{3.150}$$

$$= \sum_{i\in[F]} \pi_i(F)L_i(F) \tag{3.151}$$

$$= L(F), \tag{3.152}$$

where (A) follows from (3.141) and (3.144), and (B) follows from (2.56). □

To prove Theorem 3.1.3, for an integer $k \geq 0$, $F \in \mathscr{F}_{k\text{-dec}} \cap \mathscr{F}_{\text{ext}}$, and $i \in [F]$, we define $l_{F,i}$ as

$$l_{F,i} := \min\{|f_i^*(\boldsymbol{x}) \wedge f_i^*(\boldsymbol{x}')| : \boldsymbol{x}, \boldsymbol{x}' \in \mathcal{S}^*, f_i^*(\boldsymbol{x}) \not\preceq f_i^*(\boldsymbol{x}')\}. \tag{3.153}$$

Example 3.4.4. *Table 3.5 shows $l_{F,i}, i \in [F]$ for the code-tuples $F^{(\gamma)}, F^{(\delta)}$ and $F^{(\epsilon)}$ in Table 3.3.*

Note that

$$l_{F,i} = 0 \overset{\text{(A)}}{\iff} \mathcal{P}^1_{F,i} = \{0, 1\} \overset{\text{(B)}}{\iff} d_{F,i} = 0, \tag{3.154}$$

Table 3.5: $l_{F,i}, i \in [F]$ for the code-tuples $F^{(\gamma)}, F^{(\delta)}$ and $F^{(\epsilon)}$ in Table 3.3

F	$l_{F,0}$	$l_{F,1}$	$l_{F,2}$
$F^{(\gamma)}$	0	0	2
$F^{(\delta)}$	0	0	1
$F^{(\epsilon)}$	0	0	0

where (A) follows from (3.153), and (B) follows from (3.123).

The following Lemma 3.4.6 guarantees that the right hand side of (3.153) is well-defined.

Lemma 3.4.6. *For any integer $k \geq 0$, $F(f, \tau) \in \mathscr{F}_{k\text{-dec}} \cap \mathscr{F}_{\text{ext}}$, and $i \in [F]$, there exist $\boldsymbol{x}, \boldsymbol{x}' \in \mathcal{S}^*$ such that $f_i^*(\boldsymbol{x}) \not\succeq f_i^*(\boldsymbol{x}')$.*

Proof of Lemma 3.4.6. We prove by contradiction assuming that there exist an integer $k \geq 0$, $F(f, \tau) \in \mathscr{F}_{k\text{-dec}} \cap \mathscr{F}_{\text{ext}}$, and $i \in [F]$ such that for any $\boldsymbol{x}, \boldsymbol{x}' \in \mathcal{S}^*$, we have $f_i^*(\boldsymbol{x}) \preceq f_i^*(\boldsymbol{x}')$ or $f_i^*(\boldsymbol{x}) \succeq f_i^*(\boldsymbol{x}')$.

Choose two distinct symbols $s, s' \in \mathcal{S}$ arbitrarily. Then we have $f_i^*(s) \preceq f_i^*(s')$ or $f_i^*(s) \succeq f_i^*(s')$ by the assumption, and we may assume $f_i^*(s) \preceq f_i^*(s')$ by symmetry. By $F \in \mathscr{F}_{\text{ext}}$ and Lemma 2.3.1, we can choose $\boldsymbol{x}, \boldsymbol{x}' \in \mathcal{S}^*$ such that

$$|f_i^*(s\boldsymbol{x})| = |f_i^*(s'\boldsymbol{x}')| \geq |f_i^*(s)| + k. \tag{3.155}$$

Then by the assumption, we have

$$f_i^*(s\boldsymbol{x}) \preceq f_i^*(s'\boldsymbol{x}') \text{ or } f_i^*(s\boldsymbol{x}) \succeq f_i^*(s'\boldsymbol{x}'). \tag{3.156}$$

By (3.155) and (3.156), it holds that

$$f_i^*(s\boldsymbol{x}) = f_i^*(s'\boldsymbol{x}') \succeq f_i^*(s)\boldsymbol{c} \tag{3.157}$$

for some $\boldsymbol{c} \in \mathcal{C}^k$. By (3.157) and $s \preceq s\boldsymbol{x}$, the pair $(s, \boldsymbol{c}) \in \mathcal{S}^1 \times \mathcal{C}^k$ is not f_i^*-negative. Also, by (3.157) and $s \not\preceq s'\boldsymbol{x}$, the pair $(s, \boldsymbol{c})$ is not f_i^*-positive. Consequently, the pair $(s, \boldsymbol{c}) \in \mathcal{S}^1 \times \mathcal{C}^k \subset \mathcal{S}^* \times \mathcal{C}^k$ is neither f_i^*-positive nor f_i^*-negative. This conflicts with $F \in \mathscr{F}_{k\text{-dec}}$. □

Now we state the proof of Theorem 3.1.3 as follows.

Proof of Theorem 3.1.3. For non-negative integer $t = 0, 1, 2, \ldots$, we define $F^{(t)}(f^{(t)}, \tau^{(t)}) \in \mathscr{F}$ as follows.

$$F^{(t)} := \begin{cases} F & \text{if } t = 0, \\ \widehat{F^{(t-1)}} & \text{if } t \geq 1, \end{cases} \tag{3.158}$$

that is, $F^{(t)}$ is the code-tuple obtained by applying t times rotation to F.

From $F^{(0)} = F \in \mathscr{F}_{\mathrm{reg}} \cap \mathscr{F}_{\mathrm{ext}} \cap \mathscr{F}_{k\text{-dec}}$ and Lemmas 3.4.2–3.4.4, for any $t \geq 0$, we have $F^{(t)} \in \mathscr{F}_{\mathrm{reg}} \cap \mathscr{F}_{\mathrm{ext}} \cap \mathscr{F}_{k\text{-dec}}$ and $L(F^{(t)}) = L(F)$. Therefore, to prove Theorem 3.1.3, it suffices to prove that there exists an integer $\bar{t} \geq 0$ such that $F^{(\bar{t})} \in \mathscr{F}_{\mathrm{fork}}$. Furthermore, by (3.154), it suffices to prove that for some integer $\bar{t} \geq 0$, we have $l_{F^{(\bar{t})},0} = l_{F^{(\bar{t})},1} = \cdots = l_{F^{(\bar{t})},|F|-1} = 0$.

Fix $i \in [F]$ and choose $\boldsymbol{x}, \boldsymbol{x}' \in \mathcal{S}^*$ such that

$$f_i^{*(t)}(\boldsymbol{x}) \not\preceq f_i^{*(t)}(\boldsymbol{x}'), \tag{3.159}$$

$$|f_i^{*(t)}(\boldsymbol{x}) \wedge f_i^{*(t)}(\boldsymbol{x}')| = l_{F^{(t)},i}. \tag{3.160}$$

Then we have

$$d_{F,i} f_i^{*(t+1)}(\boldsymbol{x}) \overset{\text{(A)}}{=} f_i^{*(t)}(\boldsymbol{x}) d_{F,\tau_i^*(\boldsymbol{x})} \overset{\text{(B)}}{\not\preceq} f_i^{*(t)}(\boldsymbol{x}') d_{F,\tau_i^*(\boldsymbol{x})} \overset{\text{(C)}}{=} d_{F,i} f_i^{*(t+1)}(\boldsymbol{x}'), \tag{3.161}$$

where (A) follows from Lemma 3.4.1, (B) follows from (3.159), and (C) follows from Lemma 3.4.1. Hence, we obtain

$$f_i^{*(t+1)}(\boldsymbol{x}) \not\preceq f_i^{*(t+1)}(\boldsymbol{x}'). \tag{3.162}$$

We have

$$\begin{aligned} l_{F^{(t)},i} &\overset{\text{(A)}}{=} |f_i^{*(t)}(\boldsymbol{x}) \wedge f_i^{*(t)}(\boldsymbol{x}')| && (3.163)\\ &\overset{\text{(B)}}{=} |(f_i^{*(t)}(\boldsymbol{x}) d_{F^{(t)},\tau_i^*(\boldsymbol{x})}) \wedge (f_i^{*(t)}(\boldsymbol{x}') d_{F^{(t)},\tau_i^*(\boldsymbol{x}')})| && (3.164)\\ &\overset{\text{(C)}}{=} |(d_{F^{(t)},i} f_i^{*(t+1)}(\boldsymbol{x})) \wedge (d_{F^{(t)},i} f_i^{*(t+1)}(\boldsymbol{x}'))| && (3.165)\\ &= |d_{F^{(t)},i}| + |f_i^{*(t+1)}(\boldsymbol{x}) \wedge f_i^{*(t+1)}(\boldsymbol{x}')| && (3.166)\\ &\overset{\text{(D)}}{\geq} |d_{F^{(t)},i}| + l_{F^{(t+1)},i}, && (3.167) \end{aligned}$$

where (A) follows from (3.160), (B) follows from Lemma 3.4.1, (C) follows from (3.162), and (D) follows from (3.162). By (3.154) and (3.167), we have $l_{F^{(t+1)},i} = 0$ if $l_{F^{(t)},i} = 0$ and $l_{F^{(t+1)},i} < l_{F^{(t)},i}$ if $l_{F^{(t)},i} > 0$. Therefore, $l_{F^{(t)},i} = 0$ for any $t \geq l_{F^{(0)},i}$. Consequently, we obtain $F^{(\bar{t})} \in \mathscr{F}_{\mathrm{fork}}$, where $\bar{t} := \max\{l_{F,0}, l_{F,1}, \ldots, l_{F,|F|-1}\}$. □

3.5 Proofs of Lemmas in Chapter 3

3.5.1 Proof of Lemma 3.1.2

Proof of lemma 3.1.2. For $m \in \{1, 2, \ldots, M \coloneqq 2^{(2^k)}\}$, the number of possible tuples $(\tau_0, \tau_1, \ldots, \tau_{m-1})$ (i.e., a tuple of m mappings from $\mathcal{S}$ to $[m]$) is $m^{|\mathcal{S}|m}$, in particular, finite. Hence, the number of possible vectors $\boldsymbol{\pi}(F') = (\pi_0(F'), \pi_1(F'), \ldots, \pi_{m-1}(F'))$ of a code-tuple $F' \in \mathscr{F}'$ is also finite (cf. Remark 2.4.1), where

$$\mathscr{F}' \coloneqq \{F' \in \mathscr{F}_{\mathrm{irr}} \cap \mathscr{F}_{\mathrm{ext}} \cap \mathscr{F}_{k\text{-dec}} : |F'| \leq M\}. \tag{3.168}$$

Therefore, $\mathcal{D} \coloneqq \{\pi_i(F') : F' \in \mathscr{F}', i \in [F']\}$ is a finite set and has the minimum value $\delta \coloneqq \min \mathcal{D}$. Note that $\delta > 0$ holds since $\pi_i(F') > 0$ for any $F' \in \mathscr{F}'$ and $i \in [F']$ by $\mathscr{F}' \subseteq \mathscr{F}_{\mathrm{irr}}$ and Lemma 2.5.2 (ii).

Now, we define

$$\mathscr{F}'' \coloneqq \{F'(f', \tau') \in \mathscr{F}' : \sum_{i \in [F'], s \in \mathcal{S}} |f'_i(s)| \leq \frac{l}{\delta\nu}\}, \tag{3.169}$$

where $l \coloneqq \lceil \log_2 |\mathcal{S}| \rceil$ and $\nu \coloneqq \min_{s \in \mathcal{S}} \mu(s)$. Note that

$$0 < \nu \leq \frac{1}{|\mathcal{S}|}. \tag{3.170}$$

Then $\mathscr{F}''$ is not empty because $\tilde{F}(\tilde{f}_0, \tilde{\tau}_0) \in \mathscr{F}^{(1)}$ defined as the following (3.171) is in $\mathscr{F}''$:

$$\tilde{f}_0(s_r) = b(r), \quad \tilde{\tau}_0(s_r) = 0 \tag{3.171}$$

for $r = 0, 1, 2, \ldots, \sigma - 1$, where $\mathcal{S} = \{s_0, s_1, \ldots, s_{\sigma-1}\}$ and $b(r)$ denotes the binary representation of length l of the integer r. In fact, we obtain $\tilde{F} \in \mathscr{F}''$ by

$$\sum_{i \in [\tilde{F}], s \in \mathcal{S}} |\tilde{f}_i(s)| = \sum_{s \in \mathcal{S}} |\tilde{f}_0(s)| = |\mathcal{S}| l \overset{\text{(A)}}{\leq} \frac{l}{\nu} \overset{\text{(B)}}{\leq} \frac{l}{\delta\nu}, \tag{3.172}$$

where (A) follows from (3.170), and (B) follows from $0 < \delta \leq 1$. Since $\mathscr{F}''$ is a non-empty and finite set, there exists $F^* \in \mathscr{F}''$ such that

$$L(F^*) = \min_{F'' \in \mathscr{F}''} L(F''). \tag{3.173}$$

To complete the proof, it suffices to show that $L(F^*) \le L(F)$ for any $F \in \mathscr{F}_{\text{reg}} \cap \mathscr{F}_{\text{ext}} \cap \mathscr{F}_{k\text{-dec}}$.

First, we can see that $L(F^*) \le L(F')$ for any $F' \in \mathscr{F}'$ because for any $F'(f', \tau') \in \mathscr{F}' \setminus \mathscr{F}''$, we have

$$\begin{aligned} L(F') &= \sum_{i \in [F']} \pi_i(F') L_i(F') && (3.174) \\ &= \sum_{i \in [F']} \pi_i(F') \sum_{s \in \mathcal{S}} \mu(s) |f_i'(s)| && (3.175) \\ &\overset{\text{(A)}}{\ge} \delta\nu \sum_{i \in [F'], s \in \mathcal{S}} |f_i'(s)| && (3.176) \\ &\overset{\text{(B)}}{>} \delta\nu \cdot \frac{l}{\delta\nu} && (3.177) \\ &= l && (3.178) \\ &= L(\tilde{F}) && (3.179) \\ &\overset{\text{(C)}}{\ge} L(F^*), && (3.180) \end{aligned}$$

where (A) follows from the definitions of δ and ν, (B) follows from $F' \notin \mathscr{F}''$, and (C) follows from (3.173). Hence, we have

$$L(F^*) = \min_{F' \in \mathscr{F}'} L(F'). \tag{3.181}$$

By Theorem 3.1.1, for any $F \in \mathscr{F}_{\text{reg}} \cap \mathscr{F}_{\text{ext}} \cap \mathscr{F}_{k\text{-dec}}$, there exists $F' \in \mathscr{F}_{\text{irr}} \cap \mathscr{F}_{\text{ext}} \cap \mathscr{F}_{k\text{-dec}}$ such that $L(F') \le L(F)$ and $|\mathscr{P}^k_{F'}| = |F'|$. Then we have $F' \in \mathscr{F}'$ because

$$|F'| = |\mathscr{P}^k_{F'}| \le |\mathcal{P}(\mathcal{C}^k)| = 2^{(2^k)} = M, \tag{3.182}$$

where $\mathcal{P}(\mathcal{C}^k)$ denotes the power set of $\mathcal{C}^k$. Therefore, for any $F \in \mathscr{F}_{\text{reg}} \cap \mathscr{F}_{\text{ext}} \cap \mathscr{F}_{k\text{-dec}}$, we have

$$L(F) \ge L(F') \overset{\text{(A)}}{\ge} L(F^*) \tag{3.183}$$

as desired, where (A) follows from (3.181). □

3.5.2 Proof of Lemma 3.2.1

Proof of Lemma 3.2.1. (Proof of (i)): We prove only $\mathcal{P}^k_{F,i}(\boldsymbol{b}) \supseteq \mathcal{P}^k_{F',i}(\boldsymbol{b})$ for any $i \in [F]$ and $\boldsymbol{b} \in \mathcal{C}^*$ because we can prove $\mathcal{P}^k_{F,i}(\boldsymbol{b}) \subseteq \mathcal{P}^k_{F',i}(\boldsymbol{b})$, $\bar{\mathcal{P}}^k_{F,i}(\boldsymbol{b}) \supseteq \bar{\mathcal{P}}^k_{F',i}(\boldsymbol{b})$, and $\bar{\mathcal{P}}^k_{F,i}(\boldsymbol{b}) \subseteq \bar{\mathcal{P}}^k_{F',i}(\boldsymbol{b})$ in the similar way. To prove $\mathcal{P}^k_{F,i}(\boldsymbol{b}) \supseteq \mathcal{P}^k_{F',i}(\boldsymbol{b})$, it suffices to prove that for any $(i, \boldsymbol{x}, \boldsymbol{b}, \boldsymbol{c}) \in [F] \times \mathcal{S}^+ \times \mathcal{C}^* \times \mathcal{C}^{\leq k}$, we have

$$(f'^*_i(\boldsymbol{x}) \succeq \boldsymbol{bc}, f'_i(x_1) \succeq \boldsymbol{b}) \implies {}^\exists \boldsymbol{x}' \in \mathcal{S}^+; (f^*_i(\boldsymbol{x}') \succeq \boldsymbol{bc}, f_i(x'_1) \succeq \boldsymbol{b}) \tag{3.184}$$

because this shows that for any $i \in [F']$, $\boldsymbol{b} \in \mathcal{C}^*$, and $\boldsymbol{c} \in \mathcal{C}^k$, we have

$$\boldsymbol{c} \in \mathcal{P}^k_{F',i}(\boldsymbol{b}) \overset{\text{(A)}}{\iff} {}^\exists \boldsymbol{x} \in \mathcal{S}^+; (f'^*_i(\boldsymbol{x}) \succeq \boldsymbol{bc}, f_i(x_1) \succeq \boldsymbol{b}) \tag{3.185}$$

$$\overset{\text{(B)}}{\implies} {}^\exists \boldsymbol{x}' \in \mathcal{S}^+; (f^*_i(\boldsymbol{x}') \succeq \boldsymbol{bc}, f_i(x'_1) \succeq \boldsymbol{b}) \tag{3.186}$$

$$\overset{\text{(C)}}{\iff} \boldsymbol{c} \in \mathcal{P}^k_{F,i}(\boldsymbol{b}) \tag{3.187}$$

as desired, where (A) follows from (2.12), (B) follows from (3.184), and (C) follows from (2.12).

Choose $(i, \boldsymbol{x}, \boldsymbol{b}, \boldsymbol{c}) \in [F] \times \mathcal{S}^+ \times \mathcal{C}^* \times \mathcal{C}^{\leq k}$ arbitrarily and assume

$$f'^*_i(\boldsymbol{x}) \succeq \boldsymbol{bc} \tag{3.188}$$

and

$$f'_i(x_1) \succeq \boldsymbol{b}. \tag{3.189}$$

Then we have

$$f_i(x_1) \overset{\text{(A)}}{=} f'_i(x_1) \overset{\text{(B)}}{\succeq} \boldsymbol{b}, \tag{3.190}$$

where (A) follows from the assumption (a) of this lemma, and (B) follows from (3.189).

We prove (3.184) by induction for $|\boldsymbol{x}|$. For the base case $|\boldsymbol{x}| = 1$, we have

$$f^*_i(\boldsymbol{x}) = f_i(x_1) \overset{\text{(A)}}{=} f'_i(x_1) = f'^*_i(\boldsymbol{x}) \overset{\text{(B)}}{\succeq} \boldsymbol{bc}, \tag{3.191}$$

where (A) follows from the assumption (a) of this lemma, and (B) follows from (3.188). By (3.191) and (3.190), the claim (3.184) holds for the base case $|\boldsymbol{x}| = 1$.

We consider the induction step for $|\boldsymbol{x}| \geq 2$. We have

$$f_i(x_1) f'^*_{\tau'_i(x_1)}(\mathrm{suff}(\boldsymbol{x})) \overset{\text{(A)}}{=} f'_i(x_1) f'^*_{\tau'_i(x_1)}(\mathrm{suff}(\boldsymbol{x})) \overset{\text{(B)}}{=} f'^*_i(\boldsymbol{x}) \overset{\text{(C)}}{\succeq} \boldsymbol{bc}, \tag{3.192}$$

where (A) follows from the assumption (a) of this lemma, (B) follows from (2.4), and (C) follows from (3.188). Therefore, $f_i(x_1) \succeq \boldsymbol{bc}$ or $f_i(x_1) \prec \boldsymbol{bc}$ holds. In the case $f_i(x_1) \succeq \boldsymbol{bc}$, clearly $\boldsymbol{x}' := x_1$ satisfies $f_i^*(\boldsymbol{x}') \succeq \boldsymbol{bc}$ and $f_i(x_1') = f_i(x_1) \succeq \boldsymbol{b}$ by (3.190) as desired. Thus, now we assume $f_i(x_1) \prec \boldsymbol{bc}$. Then we have

$$|f_i(x_1)^{-1}\boldsymbol{bc}| = -|f_i(x_1)| + |\boldsymbol{b}| + |\boldsymbol{c}| \overset{\text{(A)}}{=} -|f_i'(x_1)| + |\boldsymbol{b}| + |\boldsymbol{c}| \overset{\text{(B)}}{\leq} |\boldsymbol{c}| \leq k, \tag{3.193}$$

where (A) follows from the assumption (a) of this lemma, and (B) follows from (3.189). By (3.192), we have

$$f'^*_{\tau_i'(x_1)}(\mathrm{suff}(\boldsymbol{x})) \succeq f_i(x_1)^{-1}\boldsymbol{bc}. \tag{3.194}$$

By (3.193) and (3.194), we can apply the induction hypothesis to $(\tau_i'(x_1), \mathrm{suff}(\boldsymbol{x}), \lambda, f_i(x_1)^{-1}\boldsymbol{bc})$. Hence, there exists $\boldsymbol{x}' \in \mathcal{S}^*$ such that $f^*_{\tau_i'(x_1)}(\boldsymbol{x}') \succeq f_i(x_1)^{-1}\boldsymbol{bc}$, which leads to $f_i(x_1)^{-1}\boldsymbol{bc} \in \mathcal{P}^{k'}_{F,\tau_i'(x_1)}$ by (2.16), where $k' := |f_i(x_1)^{-1}\boldsymbol{bc}|$. By Lemma 2.3.2 (i), there exists $\boldsymbol{c}' \in \mathcal{C}^{k-k'}$ such that

$$f_i(x_1)^{-1}\boldsymbol{bcc}' \in \mathcal{P}^k_{F,\tau_i'(x_1)} \overset{\text{(A)}}{=} \mathcal{P}^k_{F,\tau_i(x_1)}, \tag{3.195}$$

where (A) follows from the assumption (b) of this lemma. By (2.16), there exists $\boldsymbol{x}'' \in \mathcal{S}^*$ such that

$$f^*_{\tau_i(x_1)}(\boldsymbol{x}'') \succeq f_i(x_1)^{-1}\boldsymbol{bcc}' \succeq f_i(x_1)^{-1}\boldsymbol{bc}. \tag{3.196}$$

Thus, we have

$$f_i^*(x_1\boldsymbol{x}'') \overset{\text{(A)}}{=} f_i(x_1) f^*_{\tau_i(x_1)}(\boldsymbol{x}'') \tag{3.197}$$

$$\overset{\text{(B)}}{\succeq} f_i(x_1) f_i(x_1)^{-1}\boldsymbol{bc} \tag{3.198}$$

$$= \boldsymbol{bc}, \tag{3.199}$$

where (A) follows from (2.4), and (B) follows from (3.196). The induction is completed by (3.190) and (3.199).

(Proof of (ii)): We have

$$F \in \mathscr{F}_{\mathrm{ext}} \iff {}^\forall i \in [F]; \mathcal{P}^1_{F,i} \neq \emptyset \tag{3.200}$$

$$\overset{\text{(A)}}{\iff} {}^\forall i \in [F']; \mathcal{P}^1_{F',i} \neq \emptyset \tag{3.201}$$

$$\iff F' \in \mathscr{F}_{\mathrm{ext}}, \tag{3.202}$$

where (A) follows from (i) of this lemma.

(Proof of (iii)): For any $i \in [F']$ and $s \in \mathcal{S}$, we have

$$\mathcal{P}^k_{F',\tau'_i(s)} \cap \bar{\mathcal{P}}^k_{F',i}(f'_i(s)) \overset{\text{(A)}}{=} \mathcal{P}^k_{F,\tau'_i(s)} \cap \bar{\mathcal{P}}^k_{F,i}(f'_i(s)) \overset{\text{(B)}}{=} \mathcal{P}^k_{F,\tau_i(s)} \cap \bar{\mathcal{P}}^k_{F,i}(f_i(s)) \overset{\text{(C)}}{=} \emptyset, \tag{3.203}$$

where (A) follows from (i) of this lemma, (B) follows from the assumptions (a) and (b), and (C) follows from $F \in \mathscr{F}_{k\text{-dec}}$. Namely, F' satisfies Definition 2.2.3 (a).

For any $i \in [F']$ and $s, s' \in \mathcal{S}$ such that $s \neq s'$ and $f'_i(s) = f'_i(s')$, we have

$$f_i(s) = f_i(s') \tag{3.204}$$

by the assumption (a), and we have

$$\mathcal{P}^k_{F',\tau'_i(s)} \cap \mathcal{P}^k_{F',\tau'_i(s')} \overset{\text{(A)}}{=} \mathcal{P}^k_{F,\tau'_i(s)} \cap \mathcal{P}^k_{F,\tau'_i(s')} \overset{\text{(B)}}{=} \mathcal{P}^k_{F,\tau_i(s)} \cap \mathcal{P}^k_{F,\tau_i(s')} \overset{\text{(C)}}{=} \emptyset, \tag{3.205}$$

where (A) follows from (i) of this lemma, (B) follows from the assumptions (b), and (C) follows from $F \in \mathscr{F}_{k\text{-dec}}$ and (3.204). Namely, F' satisfies Definition 2.2.3 (b). □

3.5.3 Proof of Lemma 3.2.2

Proof of Lemma 3.2.2. We show $\mathcal{R}_{F'} \ni p$ since this implies $F' \in \mathscr{F}_{\text{reg}}$ by Lemma 2.5.2 (i). Namely, we show that for any $j \in [F']$, there exists $\boldsymbol{x} \in \mathcal{S}^*$ such that $\tau'^*_j(\boldsymbol{x}) = p$.

For $j = p$, the sequence $\boldsymbol{x} := \lambda$ satisfies $\tau'^*_j(\boldsymbol{x}) = p$ by (2.5). Thus, we now consider the case $j \neq p$. Choose $j \in [F'] \setminus \{p\}$ arbitrarily. Since $p \in \mathcal{R}_F$ by $F \in \mathscr{F}_{\text{irr}}$, there exists $\boldsymbol{x} = x_1x_2\ldots x_n \in \mathcal{S}^+$ such that $\tau^*_j(\boldsymbol{x}) = p$. Let $r \geq 1$ be the minimum positive integer such that

$$\tau^*_j(x_1x_2\ldots x_r) \in \mathcal{I}. \tag{3.206}$$

Note that there exists such an integer $r \leq n$ since $\tau^*_j(\boldsymbol{x}) = \tau^*_j(x_1x_2\ldots x_n) = p \in \mathcal{I}$. We show that

$$\tau'^*_j(x_1x_2\ldots x_{r'}) = \tau^*_j(x_1x_2\ldots x_{r'}) \tag{3.207}$$

for any $r' = 1, 2, \ldots, r-1$ by induction for r'. For the base case $r' = 0$, we have $\tau_j'^*(\lambda) = j = \tau_j^*(\lambda)$ by (2.5). We consider the induction step for $r' \geq 1$. We have

$$\tau_j'^*(x_1 x_2 \ldots x_{r'}) \overset{\text{(A)}}{=} \tau'_{\tau_j'^*(x_1 x_2 \ldots x_{r'-1})}(x_{r'}) \tag{3.208}$$

$$\overset{\text{(B)}}{=} \tau'_{\tau_j^*(x_1 x_2 \ldots x_{r'-1})}(x_{r'}) \tag{3.209}$$

$$\overset{\text{(C)}}{=} \tau_{\tau_j^*(x_1 x_2 \ldots x_{r'-1})}(x_{r'}) \tag{3.210}$$

$$\overset{\text{(D)}}{=} \tau_j^*(x_1 x_2 \ldots x_{r'}) \tag{3.211}$$

as desired, where (A) follows from Lemma 2.1.1 (ii), (B) follows from the induction hypothesis, (C) is obtained by applying the second case of (3.13) since $\tau_{\tau_j^*(x_1 x_2 \ldots x_{r'-1})}(x_{r'}) = \tau_j^*(x_1 x_2 \ldots x_{r'}) \notin \mathcal{I}$ by $r' \leq r-1$ and the minimality of r, and (D) follows from Lemma 2.1.1 (ii).

Thus, we obtain

$$\tau_j'^*(x_1 x_2 \ldots x_r) \overset{\text{(A)}}{=} \tau'_{\tau_j'^*(x_1 x_2 \ldots x_{r-1})}(x_r) \overset{\text{(B)}}{=} \tau'_{\tau_j^*(x_1 x_2 \ldots x_{r-1})}(x_r) \overset{\text{(C)}}{=} p \tag{3.212}$$

as desired, where (A) follows from Lemma 2.1.1 (ii), (B) follows from (3.207), and (C) follows from (3.206) and the first case of (3.13). □

3.5.4 Proof of Lemma 3.2.4

Proof of Lemma 3.2.4. Let $p \in \arg\min_{i \in [F]} (h_i(F) - h_i(F'))$. Then it holds that

$$^\forall i \in [F]; h_i(F') - h_p(F') \leq h_i(F) - h_p(F). \tag{3.213}$$

We have

$$\sum_{i \in [F]} (h_i(F) - h_p(F)) Q_{p,i}(F) = \sum_{i \in [F]} (h_i(F) - h_p(F)) \sum_{\substack{s \in \mathcal{S} \\ \tau_p(s) = i}} \mu(s) \tag{3.214}$$

$$= \sum_{i \in [F]} \sum_{\substack{s \in \mathcal{S} \\ \tau_p(s) = i}} (h_i(F) - h_p(F)) \mu(s) \tag{3.215}$$

$$= \sum_{i \in [F]} \sum_{\substack{s \in \mathcal{S} \\ \tau_p(s) = i}} (h_{\tau_p(s)}(F) - h_p(F)) \mu(s) \tag{3.216}$$

$$= \sum_{s \in \mathcal{S}} (h_{\tau_p(s)}(F) - h_p(F)) \mu(s). \tag{3.217}$$

Similarly, we have

$$\sum_{i\in[F]}(h_i(F)-h_p(F))Q_{p,i}(F') = \sum_{s\in\mathcal{S}}(h_{\tau'_p(s)}(F)-h_p(F))\mu(s). \tag{3.218}$$

Hence, we obtain

$$L(F') \overset{\text{(A)}}{=} L_p(F') + \sum_{i\in[F]}(h_i(F')-h_p(F'))Q_{p,i}(F') \tag{3.219}$$

$$\overset{\text{(B)}}{\leq} L_p(F') + \sum_{i\in[F]}(h_i(F)-h_p(F))Q_{p,i}(F') \tag{3.220}$$

$$\overset{\text{(C)}}{=} L_p(F') + \sum_{s\in\mathcal{S}}(h_{\tau'_p(s)}(F)-h_p(F))\mu(s) \tag{3.221}$$

$$\overset{\text{(D)}}{\leq} L_p(F) + \sum_{s\in\mathcal{S}}(h_{\tau_p(s)}(F)-h_p(F))\mu(s) \tag{3.222}$$

$$\overset{\text{(E)}}{=} L_p(F) + \sum_{i\in[F]}(h_i(F)-h_p(F))Q_{p,i}(F) \tag{3.223}$$

$$\overset{\text{(F)}}{=} L(F) \tag{3.224}$$

as desired, where (A) follows from (3.14), (B) follows from (3.213), (C) follows from (3.218), (D) follows from the assumptions (a) and (b) of this lemma, (E) follows from (3.217), and (F) follows from (3.14). □

3.5.5 Proof of Lemma 3.3.1

Proof of Lemma 3.3.1. (Proof of (i)): We prove by induction for $|\boldsymbol{z}|$. For the base case $|\boldsymbol{z}| = 0$, we have $\tau'^{*}_{\langle\lambda\rangle}(\lambda) = \langle\lambda\rangle$ by (2.5). We consider the induction step for $|\boldsymbol{z}| \geq 1$. We have

$$\tau'^{*}_{\langle\lambda\rangle}(\boldsymbol{z}) \overset{\text{(A)}}{=} \tau'_{\tau'^{*}_{\langle\lambda\rangle}(\mathrm{pref}(\boldsymbol{z}))}(z_n) \overset{\text{(B)}}{=} \tau'_{\langle\mathrm{pref}(\boldsymbol{z})\rangle}(z_n) \overset{\text{(C)}}{=} \langle\boldsymbol{z}\rangle, \tag{3.225}$$

where $\boldsymbol{z} = z_1z_2\ldots z_n$ and (A) follows from Lemma 2.1.1 (ii), (B) follows from the induction hypothesis, and (C) follows from the first case of (3.29).

(Proof of (ii)): It suffices to show that $\langle\lambda\rangle \in \mathcal{R}_{F'}$ because it guarantees that for any $j \in [F']$, there exists $\boldsymbol{x} \in \mathcal{S}^*$ such that $\tau'^{*}_j(\boldsymbol{x}) = \langle\lambda\rangle$, which leads to that for any $\boldsymbol{z} \in \mathcal{S}^{\leq L}$, we have

$$\tau'^{*}_j(\boldsymbol{x}\boldsymbol{z}) \overset{\text{(A)}}{=} \tau'^{*}_{\tau'^{*}_j(\boldsymbol{x})}(\boldsymbol{z}) = \tau'^{*}_{\langle\lambda\rangle}(\boldsymbol{z}) \overset{\text{(B)}}{=} \langle\boldsymbol{z}\rangle \tag{3.226}$$

as desired, where (A) follows from Lemma 2.1.1 (ii), and (B) follows from (i) of this lemma.

To prove $\langle\lambda\rangle \in \mathcal{R}_{F'}$, we show that there exists $\boldsymbol{x} \in \mathcal{S}^*$ such that $\tau_j'^*(\boldsymbol{x}) = \langle\lambda\rangle$ for the following two cases separately: (I) the case $j \in [F]$ and (II) the case $j = [F'] \setminus [F]$.

(I) The case $j \in [F]$: By the assumption that $p = \langle\lambda\rangle \in \mathcal{R}_F$, there exists $\boldsymbol{x} = x_1x_2\ldots x_{n'} \in \mathcal{S}^*$ such that $\tau_j^*(\boldsymbol{x}) = \langle\lambda\rangle$. We choose the shortest $\boldsymbol{x}$ among such sequences. Then we can see $\tau_j'^*(x_1x_2\ldots x_r) = \tau_j^*(x_1x_2\ldots x_r)$ for any $r = 0,1,2,\ldots,n$ by induction for r. For the base case $r = 0$, we have $\tau_j'^*(\lambda) = j = \tau_j^*(\lambda)$ by (2.5). We consider the induction step for $r \geq 1$. We have

$$\tau_j'^*(x_1x_2\ldots x_r) \stackrel{\text{(A)}}{=} \tau'_{\tau_j'^*(x_1x_2\ldots x_{r-1})}(x_r) \tag{3.227}$$

$$\stackrel{\text{(B)}}{=} \tau'_{\tau_j^*(x_1x_2\ldots x_{r-1})}(x_r) \tag{3.228}$$

$$\stackrel{\text{(C)}}{=} \tau_{\tau_j^*(x_1x_2\ldots x_{r-1})}(x_r) \tag{3.229}$$

$$\stackrel{\text{(D)}}{=} \tau_j^*(x_1x_2\ldots x_r) \tag{3.230}$$

as desired, where (A) follows from (2.5), (B) follows from the induction hypothesis, (C) follows from the third case of (3.29) since $\tau_j^*(x_1x_2\ldots x_{r-1}) \in [F] \setminus \{\langle\lambda\rangle\}$ by the definition of $\boldsymbol{x}$, and (D) follows from Lemma 2.1.1 (ii). Therefore, we obtain $\tau_j'^*(\boldsymbol{x}) = \tau_j^*(\boldsymbol{x}) = \langle\lambda\rangle$ as desired.

(II) The case where $j = [F'] \setminus [F]$: Then we have $j = \langle\boldsymbol{z}\rangle$ for some $\boldsymbol{z} \in \mathcal{S}^{\leq L}$. Choose $\boldsymbol{z'} = z_1'z_2'\ldots z_{n'}' \in \mathcal{S}^{L-|\boldsymbol{z}|+1}$ arbitrarily. We have

$$\tau_{\langle\lambda\rangle}'^*(\boldsymbol{z}\boldsymbol{z'}) \stackrel{\text{(A)}}{=} \tau'_{\tau_{\langle\lambda\rangle}'^*(\boldsymbol{z}\mathrm{pref}(\boldsymbol{z'}))}(z_{n'}') \tag{3.231}$$

$$\stackrel{\text{(B)}}{=} \tau'_{\langle\boldsymbol{z}\mathrm{pref}(\boldsymbol{z'})\rangle}(z_{n'}') \stackrel{\text{(C)}}{=} \tau_{\langle\lambda\rangle}^*(\boldsymbol{z}\boldsymbol{z'}) \tag{3.232}$$

$$= \tau_{|F|-1}^*(\boldsymbol{z}\boldsymbol{z'}) \in [F], \tag{3.233}$$

where (A) follows from Lemma 2.1.1 (ii), (B) follows from (i) of this lemma and $\boldsymbol{z}\mathrm{pref}(\boldsymbol{z'}) \in \mathcal{S}^{\leq L}$, and (C) follows from the second case of (3.29) and $\boldsymbol{z}\mathrm{pref}(\boldsymbol{z'}) \in \mathcal{S}^L$. Hence, by the discussion for the case (I) above, there exists $\boldsymbol{x}' \in \mathcal{S}^*$ such that $\tau'^*_{\tau'^*_{\langle\lambda\rangle}(\boldsymbol{z}\boldsymbol{z}')}(\boldsymbol{x}') = \langle\lambda\rangle$. Thus, $\boldsymbol{x} \coloneqq \boldsymbol{z}'\boldsymbol{x}'$ satisfies

$$\tau'^*_{\langle\boldsymbol{z}\rangle}(\boldsymbol{x}) = \tau'^*_{\langle\boldsymbol{z}\rangle}(\boldsymbol{z}'\boldsymbol{x}') \overset{\text{(A)}}{=} \tau'^*_{\tau'^*_{\langle\lambda\rangle}(\boldsymbol{z})}(\boldsymbol{z}'\boldsymbol{x}') \overset{\text{(B)}}{=} \tau'^*_{\langle\lambda\rangle}(\boldsymbol{z}\boldsymbol{z}'\boldsymbol{x}') \overset{\text{(C)}}{=} \tau'^*_{\tau'^*_{\langle\lambda\rangle}(\boldsymbol{z}\boldsymbol{z}')}(\boldsymbol{x}') = \langle\lambda\rangle, \tag{3.234}$$

where (A) follows from (i) of this lemma, (B) follows from Lemma 2.1.1 (ii), and (C) follows from Lemma 2.1.1 (ii).

□

3.5.6 Proof of Lemma 3.3.2

Proof of Lemma 3.3.2. (Proof of (i)): We prove by the induction for $|\boldsymbol{x}|$. For the base case $|\boldsymbol{x}| = 0$, we have $f''^*_{\langle\boldsymbol{z}\rangle}(\lambda) = \lambda = f'^*_{\langle\boldsymbol{z}\rangle}(\lambda)$ by (2.4). We consider the induction step for $|\boldsymbol{x}| \geq 1$ choosing $\boldsymbol{z} \in \mathcal{S}^{\leq L}$ arbitrarily and dividing into the following two cases: the case $f'^*_{\langle\lambda\rangle}(\boldsymbol{z}) \prec \boldsymbol{d} \preceq f'^*_{\langle\lambda\rangle}(\boldsymbol{z}\boldsymbol{x})$ and the other case.

- The case $f'^*_{\langle\lambda\rangle}(\boldsymbol{z}) \prec \boldsymbol{d} \preceq f'^*_{\langle\lambda\rangle}(\boldsymbol{z}\boldsymbol{x})$: We consider the following two cases separately: the case $f'^*_{\langle\lambda\rangle}(\boldsymbol{z}) \prec \boldsymbol{d} \preceq f'^*_{\langle\lambda\rangle}(\boldsymbol{z}x_1)$ and the case $f'^*_{\langle\lambda\rangle}(\boldsymbol{z}x_1) \prec \boldsymbol{d} \preceq f'^*_{\langle\lambda\rangle}(\boldsymbol{z}\boldsymbol{x})$.
  - The case $f'^*_{\langle\lambda\rangle}(\boldsymbol{z}) \prec \boldsymbol{d} \preceq f'^*_{\langle\lambda\rangle}(\boldsymbol{z}x_1)$: We have

$$f''^*_{\langle\boldsymbol{z}\rangle}(\boldsymbol{x}) \overset{\text{(A)}}{=} f''_{\langle\boldsymbol{z}\rangle}(x_1) f''^*_{\langle\boldsymbol{z}x_1\rangle}(\mathrm{suff}(\boldsymbol{x})) \tag{3.235}$$

$$\overset{\text{(B)}}{=} f'^*_{\langle\boldsymbol{z}\rangle}(\boldsymbol{z})^{-1}\mathrm{pref}(\boldsymbol{d})\boldsymbol{d}^{-1} f'^*_{\langle\lambda\rangle}(\boldsymbol{z}x_1) f''^*_{\langle\boldsymbol{z}x_1\rangle}(\mathrm{suff}(\boldsymbol{x})) \tag{3.236}$$

$$\overset{\text{(C)}}{=} f'^*_{\langle\boldsymbol{z}\rangle}(\boldsymbol{z})^{-1}\mathrm{pref}(\boldsymbol{d})\boldsymbol{d}^{-1} f'^*_{\langle\lambda\rangle}(\boldsymbol{z}x_1) f'^*_{\langle\boldsymbol{z}x_1\rangle}(\mathrm{suff}(\boldsymbol{x})) \tag{3.237}$$

$$\overset{\text{(D)}}{=} f'^*_{\langle\boldsymbol{z}\rangle}(\boldsymbol{z})^{-1}\mathrm{pref}(\boldsymbol{d})\boldsymbol{d}^{-1} f'^*_{\langle\lambda\rangle}(\boldsymbol{z}\boldsymbol{x}), \tag{3.238}$$

  where (A) follows from (2.4) and Lemma 3.3.1 (i), (B) follows from the first case of (3.35) and $f'^*_{\langle\lambda\rangle}(\boldsymbol{z}) \prec \boldsymbol{d} \preceq f'^*_{\langle\lambda\rangle}(\boldsymbol{z}x_1)$, (C) follows from the second case of (3.37) by the induction hypothesis and $f'^*_{\langle\lambda\rangle}(\boldsymbol{z}x_1) \not\prec \boldsymbol{d}$, and (D) follows from (2.4).

– The case $f'^*_{\langle\lambda\rangle}(\boldsymbol{z}x_1) \prec \boldsymbol{d} \preceq f'^*_{\langle\lambda\rangle}(\boldsymbol{zx})$: We have

$$f''^*_{\langle\boldsymbol{z}\rangle}(\boldsymbol{x}) \overset{\text{(A)}}{=} f''_{\langle\boldsymbol{z}\rangle}(x_1) f''^*_{\langle\boldsymbol{z}x_1\rangle}(\text{suff}(\boldsymbol{x})) \tag{3.239}$$

$$\overset{\text{(B)}}{=} f'_{\langle\boldsymbol{z}\rangle}(x_1) f''^*_{\langle\boldsymbol{z}x_1\rangle}(\text{suff}(\boldsymbol{x})) \tag{3.240}$$

$$\overset{\text{(C)}}{=} f'_{\langle\boldsymbol{z}\rangle}(x_1) f'^*_{\langle\lambda\rangle}(\boldsymbol{z}x_1)^{-1} \text{pref}(\boldsymbol{d})\boldsymbol{d}^{-1}(f'^*_{\langle\lambda\rangle}(\boldsymbol{zx})) \tag{3.241}$$

$$\overset{\text{(D)}}{=} f'^*_{\langle\boldsymbol{z}\rangle}(\boldsymbol{z})^{-1} \text{pref}(\boldsymbol{d})\boldsymbol{d}^{-1} f'^*_{\langle\lambda\rangle}(\boldsymbol{zx}), \tag{3.242}$$

where (A) follows from (2.4) and Lemma 3.3.1 (i), (B) follows from the second case of (3.35) since $\boldsymbol{d} \not\preceq f'^*_{\langle\lambda\rangle}(\boldsymbol{z}x_1)$, (C) follows from the first case of (3.37) by the induction hypothesis and $f'^*_{\langle\lambda\rangle}(\boldsymbol{z}x_1) \prec \boldsymbol{d} \preceq f'^*_{\langle\lambda\rangle}(\boldsymbol{zx})$, and (D) follows from (2.4).

- The other case: We have

$$f''^*_{\langle\boldsymbol{z}\rangle}(\boldsymbol{x}) \overset{\text{(A)}}{=} f''_{\langle\boldsymbol{z}\rangle}(x_1) f''^*_{\langle\boldsymbol{z}x_1\rangle}(\text{suff}(\boldsymbol{x})) \tag{3.243}$$

$$\overset{\text{(B)}}{=} f'_{\langle\boldsymbol{z}\rangle}(x_1) f''^*_{\langle\boldsymbol{z}x_1\rangle}(\text{suff}(\boldsymbol{x})) \tag{3.244}$$

$$\overset{\text{(C)}}{=} f'_{\langle\boldsymbol{z}\rangle}(x_1) f'^*_{\langle\boldsymbol{z}x_1\rangle}(\text{suff}(\boldsymbol{x})) \tag{3.245}$$

$$\overset{\text{(D)}}{=} f'^*_{\langle\boldsymbol{z}\rangle}(\boldsymbol{x}), \tag{3.246}$$

where (A) follows from (2.4) and Lemma 3.3.1 (i), (B) follows from the second case of (3.35) since $f'^*_{\langle\lambda\rangle}(\boldsymbol{z}) \prec \boldsymbol{d} \preceq f'^*_{\langle\lambda\rangle}(\boldsymbol{z}x_1)$ does not hold, (C) follows from the second case of (3.37) by the induction hypothesis and that $f'^*_{\langle\lambda\rangle}(\boldsymbol{z}) \prec \boldsymbol{d} \preceq f'^*_{\langle\lambda\rangle}(\boldsymbol{z}x_1)$ does not hold, and (D) follows from (2.4).

(Proof of (ii)): Assume that

$$f''_{\langle\boldsymbol{z}\rangle}(s) \prec f''_{\langle\boldsymbol{z}\rangle}(s'). \tag{3.247}$$

In the case $f'^*_{\langle\lambda\rangle}(\boldsymbol{z}) \not\prec \boldsymbol{d}$, we have

$$f'_{\langle\boldsymbol{z}\rangle}(s) \overset{\text{(A)}}{=} f''_{\langle\boldsymbol{z}\rangle}(s) \overset{\text{(B)}}{\prec} f''_{\langle\boldsymbol{z}\rangle}(s') \overset{\text{(C)}}{=} f'_{\langle\boldsymbol{z}\rangle}(s') \tag{3.248}$$

as desired, where (A) follows from the second case of (3.35) and $f'^*_{\langle\lambda\rangle}(\boldsymbol{z}) \not\prec \boldsymbol{d}$, (B) follows from (3.247), and (C) follows from the second case of (3.35) and $f'^*_{\langle\lambda\rangle}(\boldsymbol{z}) \not\prec \boldsymbol{d}$.

We consider the case $f'^*_{\langle\lambda\rangle}(\boldsymbol{z}) \prec \boldsymbol{d}$ dividing into four cases by whether $\boldsymbol{d} \preceq f'^*_{\langle\lambda\rangle}(\boldsymbol{z}s)$ and whether $\boldsymbol{d} \preceq f'^*_{\langle\lambda\rangle}(\boldsymbol{z}s')$.

- The case $\boldsymbol{d} \preceq f'^*_{\langle\lambda\rangle}(\boldsymbol{z}s), \boldsymbol{d} \preceq f'^*_{\langle\lambda\rangle}(\boldsymbol{z}s')$: We have

$$
\begin{aligned}
& f'^*_{\langle\lambda\rangle}(\boldsymbol{z})^{-1}\mathrm{pref}(\boldsymbol{d})\boldsymbol{d}^{-1}(f'^*_{\langle\lambda\rangle}(\boldsymbol{z})f'^*_{\langle\boldsymbol{z}\rangle}(s)) && (3.249)\\
& \overset{\text{(A)}}{=} f'^*_{\langle\lambda\rangle}(\boldsymbol{z})^{-1}\mathrm{pref}(\boldsymbol{d})\boldsymbol{d}^{-1}f'^*_{\langle\lambda\rangle}(\boldsymbol{z}s) && (3.250)\\
& \overset{\text{(B)}}{=} f''_{\langle\boldsymbol{z}\rangle}(s) && (3.251)\\
& \overset{\text{(C)}}{\prec} f''_{\langle\boldsymbol{z}\rangle}(s') && (3.252)\\
& \overset{\text{(D)}}{=} f'^*_{\langle\lambda\rangle}(\boldsymbol{z})^{-1}\mathrm{pref}(\boldsymbol{d})\boldsymbol{d}^{-1}f'^*_{\langle\lambda\rangle}(\boldsymbol{z}s') && (3.253)\\
& \overset{\text{(E)}}{=} f'^*_{\langle\lambda\rangle}(\boldsymbol{z})^{-1}\mathrm{pref}(\boldsymbol{d})\boldsymbol{d}^{-1}(f'^*_{\langle\lambda\rangle}(\boldsymbol{z})f'_{\langle\boldsymbol{z}\rangle}(s')), && (3.254)
\end{aligned}
$$

where (A) follows from Lemma 2.1.1 (i) and Lemma 3.3.1 (i), (B) follows from the first case of (3.35) and $\boldsymbol{d} \preceq f'^*_{\langle\lambda\rangle}(\boldsymbol{z}s)$, (C) follows from (3.247), (D) follows from the first case of (3.35) and $\boldsymbol{d} \preceq f'^*_{\langle\lambda\rangle}(\boldsymbol{z}s')$, and (E) follows from Lemma 2.1.1 (i) and Lemma 3.3.1 (i). Comparing both sides of (3.254), we obtain $f'_{\langle\boldsymbol{z}\rangle}(s) \prec f'_{\langle\boldsymbol{z}\rangle}(s')$ as desired.

- The case $\boldsymbol{d} \preceq f'^*_{\langle\lambda\rangle}(\boldsymbol{z}s), \boldsymbol{d} \not\preceq f'^*_{\langle\lambda\rangle}(\boldsymbol{z}s')$: We show that this case is impossible. We have

$$
\begin{aligned}
f'^*_{\langle\lambda\rangle}(\boldsymbol{z}s') & \overset{\text{(A)}}{=} f'^*_{\langle\lambda\rangle}(\boldsymbol{z})f'_{\langle\boldsymbol{z}\rangle}(s') && (3.255)\\
& \overset{\text{(B)}}{=} f'^*_{\langle\lambda\rangle}(\boldsymbol{z})f''_{\langle\boldsymbol{z}\rangle}(s') && (3.256)\\
& \overset{\text{(C)}}{\succ} f'^*_{\langle\lambda\rangle}(\boldsymbol{z})f''_{\langle\boldsymbol{z}\rangle}(s) && (3.257)\\
& \overset{\text{(D)}}{=} f'^*_{\langle\lambda\rangle}(\boldsymbol{z})f'^*_{\langle\lambda\rangle}(\boldsymbol{z})^{-1}\boldsymbol{d}\mathrm{pref}(\boldsymbol{d})^{-1}f'^*_{\langle\lambda\rangle}(\boldsymbol{z}s) && (3.258)\\
& = \boldsymbol{d}\mathrm{pref}(\boldsymbol{d})^{-1}\boldsymbol{d} && (3.259)\\
& \succeq \boldsymbol{d}, && (3.260)
\end{aligned}
$$

where (A) follows from Lemma 2.1.1 (i) and Lemma 3.3.1 (i), (B) follows from the second case of (3.35) and $\boldsymbol{d} \not\preceq f'^*_{\langle\lambda\rangle}(\boldsymbol{z}s')$, (C) follows from (3.247), and (D) follows from the first case of (3.35) and $\boldsymbol{d} \preceq f'^*_{\langle\lambda\rangle}(\boldsymbol{z}s)$. This conflicts with $\boldsymbol{d} \not\preceq f'^*_{\langle\lambda\rangle}(\boldsymbol{z}s')$.

- The case $\boldsymbol{d} \not\preceq f'^*_{\langle\lambda\rangle}(\boldsymbol{z}s), \boldsymbol{d} \preceq f'^*_{\langle\lambda\rangle}(\boldsymbol{z}s')$: We have

$$f'^*_{\langle\lambda\rangle}(\boldsymbol{z}s) \overset{\text{(A)}}{=} f'^*_{\langle\lambda\rangle}(\boldsymbol{z})f'_{\langle\boldsymbol{z}\rangle}(s) \tag{3.261}$$

$$\overset{\text{(B)}}{=} f'^*_{\langle\lambda\rangle}(\boldsymbol{z})f''_{\langle\boldsymbol{z}\rangle}(s) \tag{3.262}$$

$$\overset{\text{(C)}}{\prec} f'^*_{\langle\lambda\rangle}(\boldsymbol{z})f''_{\langle\boldsymbol{z}\rangle}(s') \tag{3.263}$$

$$\overset{\text{(D)}}{=} f'^*_{\langle\lambda\rangle}(\boldsymbol{z})f'^*_{\langle\lambda\rangle}(\boldsymbol{z})^{-1}\boldsymbol{d}\mathrm{pref}(\boldsymbol{d})^{-1}f'^*_{\langle\lambda\rangle}(\boldsymbol{z}s') \tag{3.264}$$

$$= \boldsymbol{d}\mathrm{pref}(\boldsymbol{d})^{-1}f'^*_{\langle\lambda\rangle}(\boldsymbol{z}s'), \tag{3.265}$$

where (A) follows from Lemma 2.1.1 (i) and Lemma 3.3.1 (i), (B) follows from the second case of (3.35) and $\boldsymbol{d} \not\preceq f'^*_{\langle\lambda\rangle}(\boldsymbol{z}s)$, (C) follows from (3.247), and (D) follows from the first case of (3.35) and $\boldsymbol{d} \preceq f'^*_{\langle\lambda\rangle}(\boldsymbol{z}s')$. Therefore, we have at least one of $f'^*_{\langle\lambda\rangle}(\boldsymbol{z}s) \prec \boldsymbol{d}$ and $f'^*_{\langle\lambda\rangle}(\boldsymbol{z}s) \succeq \boldsymbol{d}$. Since $\boldsymbol{d} \not\preceq f'^*_{\langle\lambda\rangle}(\boldsymbol{z}s)$, we have $f'^*_{\langle\lambda\rangle}(\boldsymbol{z}s) \prec \boldsymbol{d}$. Thus, we have $f'^*_{\langle\lambda\rangle}(\boldsymbol{z}s) \prec \boldsymbol{d} \preceq f'^*_{\langle\lambda\rangle}(\boldsymbol{z}s')$, which leads to $f'_{\langle\boldsymbol{z}\rangle}(s) \prec f'_{\langle\boldsymbol{z}\rangle}(s')$ as desired.

- The case $\boldsymbol{d} \not\preceq f'^*_{\langle\lambda\rangle}(\boldsymbol{z}s), \boldsymbol{d} \not\preceq f'^*_{\langle\lambda\rangle}(\boldsymbol{z}s')$: We have

$$f'_{\langle\boldsymbol{z}\rangle}(s) \overset{\text{(A)}}{=} f''_{\langle\boldsymbol{z}\rangle}(s) \overset{\text{(B)}}{\prec} f''_{\langle\boldsymbol{z}\rangle}(s') \overset{\text{(C)}}{=} f'_{\langle\boldsymbol{z}\rangle}(s') \tag{3.266}$$

as desired, where (A) follows from the second case of (3.35) and $\boldsymbol{d} \not\preceq f'^*_{\langle\lambda\rangle}(\boldsymbol{z}s)$, (B) follows from (3.247), and (C) follows from the second case of (3.35) and $\boldsymbol{d} \not\preceq f'^*_{\langle\lambda\rangle}(\boldsymbol{z}s')$.

(Proof of (iii)): Choose $\boldsymbol{x} \in \mathcal{S}^{\geq L}$ arbitrarily. We have

$$|f'^*_{\langle\lambda\rangle}(\boldsymbol{x})| \overset{\text{(A)}}{=} |f^*_{\langle\lambda\rangle}(\boldsymbol{x})| \overset{\text{(B)}}{\geq} \left\lfloor \frac{|\boldsymbol{x}|}{|F|} \right\rfloor \geq \left\lfloor \frac{L}{|F|} \right\rfloor \overset{\text{(C)}}{=} \left\lfloor \frac{|F|(|\boldsymbol{d}|+1)}{|F|} \right\rfloor = |\boldsymbol{d}| + 1, \tag{3.267}$$

where (A) follows from Lemma 2.5.1 (i) since φ defined in (3.30) is a homomorphism from F' to F, (B) follows from Lemma 2.3.3, and (C) follows from the definition of L. Also, we have

$$|f''^*_{\langle\lambda\rangle}(\boldsymbol{x})| \overset{\text{(A)}}{\geq} \min\{|f'^*_{\langle\lambda\rangle}(\boldsymbol{x})|, |f'^*_{\langle\lambda\rangle}(\boldsymbol{z})^{-1}\mathrm{pref}(\boldsymbol{d})\boldsymbol{d}^{-1}(f'^*_{\langle\lambda\rangle}(\boldsymbol{z}\boldsymbol{x}))|\} \tag{3.268}$$

$$= \min\{|f'^*_{\langle\boldsymbol{z}\rangle}(\boldsymbol{x})|, |f'^*_{\langle\boldsymbol{z}\rangle}(\boldsymbol{x})| - 1\} \tag{3.269}$$

$$\overset{\text{(B)}}{\geq} |\boldsymbol{d}|, \tag{3.270}$$

where (A) follows from (i) of this lemma, and (B) follows from (3.267). □

3.5.7 Proof of Lemma 3.3.3

Proof of Lemma 3.3.3. (Proof of (i)): Assume

$$f''^*_{\langle\lambda\rangle}(\boldsymbol{x}) \succeq \boldsymbol{c}. \tag{3.271}$$

We consider the following two cases separately: the case $\boldsymbol{d} \preceq f'^*_{\langle\lambda\rangle}(\boldsymbol{x})$ and the case $\boldsymbol{d} \not\preceq f'^*_{\langle\lambda\rangle}(\boldsymbol{x})$.

- The case $\boldsymbol{d} \preceq f'^*_{\langle\lambda\rangle}(\boldsymbol{x})$: We have

$$f''^*_{\langle\lambda\rangle}(\boldsymbol{x}) \overset{\text{(A)}}{=} \mathrm{pref}(\boldsymbol{d})\boldsymbol{d}^{-1} f'^*_{\langle\lambda\rangle}(\boldsymbol{x}) \succeq \mathrm{pref}(\boldsymbol{d}), \tag{3.272}$$

  where (A) follows from the first case of (3.37) and $\boldsymbol{d} \preceq f'^*_{\langle\lambda\rangle}(\boldsymbol{x})$. Comparing (3.271) and (3.272), we have $\mathrm{pref}(\boldsymbol{d}) \succeq \boldsymbol{c}$ since $|\mathrm{pref}(\boldsymbol{d})| \geq k \geq |\boldsymbol{c}|$. Therefore, by $\boldsymbol{d} \preceq f'^*_{\langle\lambda\rangle}(\boldsymbol{x})$, we obtain $f'^*_{\langle\lambda\rangle}(\boldsymbol{x}) \succeq \boldsymbol{d} \succeq \mathrm{pref}(\boldsymbol{d}) \succeq \boldsymbol{c}$ as desired.

- The case $\boldsymbol{d} \not\preceq f'^*_{\langle\lambda\rangle}(\boldsymbol{x})$: We have

$$f'^*_{\langle\lambda\rangle}(\boldsymbol{x}) \overset{\text{(A)}}{=} f''^*_{\langle\lambda\rangle}(\boldsymbol{x}) \overset{\text{(B)}}{\succeq} \boldsymbol{c}, \tag{3.273}$$

  where (A) follows from the second case of (3.37) and $\boldsymbol{d} \not\preceq f'^*_{\langle\lambda\rangle}(\boldsymbol{x})$, and (B) follows from (3.271).

(Proof of (ii)): For $i \in [F] \setminus \{\langle\lambda\rangle\}$, we have $f''_i(s) = f'_i(s)$ directly from the second case of (3.35). We consider the case where $i = \langle\boldsymbol{z}\rangle$ for some $\boldsymbol{z} \in \mathcal{S}^L$. Then we have $f'^*_{\langle\lambda\rangle}(\boldsymbol{z}) \not\preceq \boldsymbol{d}$ because $|f'^*_{\langle\lambda\rangle}(\boldsymbol{z})| \geq |\boldsymbol{d}| + 1$ by Lemma 3.3.2 (iii). Therefore, by the second case of (3.35), we obtain $f''_i(s) = f'_i(s)$.

(Proof of (iii)): We prove only that $\mathcal{P}^k_{F'',i}(\boldsymbol{b}) \subseteq \mathcal{P}^k_{F',i}(\boldsymbol{b})$ for any $i \in \mathcal{J}$ and $\boldsymbol{b} \in \mathcal{C}^*$ because we can prove $\bar{\mathcal{P}}^k_{F'',i}(\boldsymbol{b}) \subseteq \bar{\mathcal{P}}^k_{F',i}(\boldsymbol{b})$ in the similar way. To prove $\mathcal{P}^k_{F'',i}(\boldsymbol{b}) \subseteq \mathcal{P}^k_{F',i}(\boldsymbol{b})$, it suffices to prove that for any $(i, \boldsymbol{x}, \boldsymbol{b}, \boldsymbol{c}) \in \mathcal{J} \times \mathcal{S}^+ \times \mathcal{C}^* \times \mathcal{C}^{\leq k}$, we have

$$(f''^*_i(\boldsymbol{x}) \succeq \boldsymbol{bc}, f''_i(x_1) \succeq \boldsymbol{b}) \implies {}^\exists \boldsymbol{x}' \in \mathcal{S}^+; (f'^*_i(\boldsymbol{x}') \succeq \boldsymbol{bc}, f'_i(x'_1) \succeq \boldsymbol{b}) \tag{3.274}$$

because this shows that for any $i \in \mathcal{J}$, $\boldsymbol{b} \in \mathcal{C}^*$, and $\boldsymbol{c} \in \mathcal{C}^k$, we have

$$\begin{aligned}
\boldsymbol{c} \in \mathcal{P}^k_{F'',i}(\boldsymbol{b}) &\overset{\text{(A)}}{\iff} {}^\exists \boldsymbol{x} \in \mathcal{S}^+; (f''^*_i(\boldsymbol{x}) \succeq \boldsymbol{bc}, f''_i(x_1) \succeq \boldsymbol{b}) && (3.275)\\
&\overset{\text{(B)}}{\implies} {}^\exists \boldsymbol{x}' \in \mathcal{S}^+; (f'^*_i(\boldsymbol{x}') \succeq \boldsymbol{bc}, f'_i(x'_1) \succeq \boldsymbol{b}) && (3.276)\\
&\overset{\text{(C)}}{\iff} \boldsymbol{c} \in \mathcal{P}^k_{F',i}(\boldsymbol{b}) && (3.277)
\end{aligned}$$

as desired, where (A) follows from (2.12), (B) follows from (3.274), and (C) follows from (2.12).

Choose $(i, \boldsymbol{x}, \boldsymbol{b}, \boldsymbol{c}) \in [F] \times \mathcal{S}^+ \times \mathcal{C}^* \times \mathcal{C}^{\leq k}$ arbitrarily and assume

$$f_i''^*(\boldsymbol{x}) \succeq \boldsymbol{bc} \tag{3.278}$$

and

$$f_i''(x_1) \succeq \boldsymbol{b}. \tag{3.279}$$

Then we have

$$f_i'(x_1) \overset{\text{(A)}}{=} f_i''(x_1) \overset{\text{(B)}}{\succeq} \boldsymbol{b}, \tag{3.280}$$

where (A) follows from (ii) of this lemma, and (B) follows from (3.279).

We prove (3.274) by induction for $|\boldsymbol{x}|$. For the base case $|\boldsymbol{x}| = 1$, we have

$$f_i'^*(\boldsymbol{x}) = f_i'(x_1) \overset{\text{(A)}}{=} f_i''(x_1) = f_i''^*(\boldsymbol{x}) \overset{\text{(B)}}{\succeq} \boldsymbol{bc} \tag{3.281}$$

as desired, where (A) follows from (ii) of this lemma, and (B) follows from (3.278). By (3.281) and (3.280), the claim (3.274) holds for the base case $|\boldsymbol{x}| = 1$.

We consider the induction step for $|\boldsymbol{x}| \geq 2$. We have

$$f_i'(x_1) f''^*_{\tau_i''(x_1)}(\mathrm{suff}(\boldsymbol{x})) \overset{\text{(A)}}{=} f_i''(x_1) f''^*_{\tau_i''(x_1)}(\mathrm{suff}(\boldsymbol{x})) \overset{\text{(B)}}{=} f_i''^*(\boldsymbol{x}) \overset{\text{(C)}}{\succeq} \boldsymbol{bc}, \tag{3.282}$$

where (A) follows from (ii) of this lemma, (B) follows from (2.4), and (C) follows from (3.278).

Therefore, $f_i'(x_1) \succeq \boldsymbol{bc}$ or $f_i'(x_1) \prec \boldsymbol{bc}$ holds. In the case $f_i'(x_1) \succeq \boldsymbol{bc}$, the sequence $\boldsymbol{x}' \coloneqq x_1$ satisfies $f_i'^*(\boldsymbol{x}') \succeq \boldsymbol{bc}$ and $f_i'(x_1') = f_i'(x_1) \succeq \boldsymbol{b}$ by (3.280) as desired. Thus, now we assume $f_i'(x_1) \prec \boldsymbol{bc}$. Then we have

$$|f_i'(x_1)^{-1}\boldsymbol{bc}| = -|f_i'(x_1)| + |\boldsymbol{b}| + |\boldsymbol{c}| \overset{\text{(A)}}{=} -|f_i''(x_1)| + |\boldsymbol{b}| + |\boldsymbol{c}| \overset{\text{(B)}}{\leq} |\boldsymbol{c}| \leq k, \tag{3.283}$$

where (A) follows from (ii) of this lemma, and (B) follows from (3.279). By (3.282), we have

$$f''^*_{\tau_i''(x_1)}(\mathrm{suff}(\boldsymbol{x})) \succeq f_i'(x_1)^{-1}\boldsymbol{bc}. \tag{3.284}$$

We can see that there exists $\boldsymbol{x}' \in \mathcal{S}^+$ such that

$$f'^*_{\tau_i''(x_1)}(\boldsymbol{x}') \succeq f_i'(x_1)^{-1}\boldsymbol{bc} \tag{3.285}$$

as follows.

- The case $\tau_i''(x_1) = \langle\lambda\rangle$: By (3.283), we can apply (i) of this lemma to obtain that $\boldsymbol{x}' \coloneqq \mathrm{suff}(\boldsymbol{x})$ satisfies (3.285) from (3.284).
- The case $\tau_i''(x_1) \in \mathcal{J}$: By (3.283) and (3.284), we can apply the induction hypothesis to $(\tau_i''(x_1), \mathrm{suff}(\boldsymbol{x}), \lambda, f_i'(x_1)^{-1}\boldsymbol{bc})$.

Therefore, we have

$$f_i'^{*}(x_1\boldsymbol{x}') \overset{\text{(A)}}{=} f_i'(x_1) f'^{*}_{\tau_i'(x_1)}(\boldsymbol{x}') \overset{\text{(B)}}{=} f_i'(x_1) f'^{*}_{\tau_i''(x_1)}(\boldsymbol{x}') \overset{\text{(C)}}{\succeq} f_i'(x_1) f_i'(x_1)^{-1}\boldsymbol{bc} = \boldsymbol{bc}, \tag{3.286}$$

where (A) follows from (2.4), (B) follows from (3.36), and (C) follows from (3.285). The induction is completed by (3.280) and (3.286). □

3.5.8 Proof of Lemma 3.3.4

Proof of Lemma 3.3.4. (Proof of (i)): Assume that

$$\boldsymbol{b} \preceq \boldsymbol{b}'. \tag{3.287}$$

In the case $f'^{*}_{\langle\lambda\rangle}(\boldsymbol{z}) \not\preceq \mathrm{pref}(\boldsymbol{d})$, we have

$$\psi_{\boldsymbol{z}}(\boldsymbol{b}) \overset{\text{(A)}}{=} \boldsymbol{b} \overset{\text{(B)}}{\preceq} \boldsymbol{b}' \overset{\text{(C)}}{=} \psi_{\boldsymbol{z}}(\boldsymbol{b}'), \tag{3.288}$$

where (A) follows from the second case of (3.58) and $f'^{*}_{\langle\lambda\rangle}(\boldsymbol{z}) \not\preceq \mathrm{pref}(\boldsymbol{d})$, (B) follows from (3.287), and (C) follows from the second case of (3.58) and $f'^{*}_{\langle\lambda\rangle}(\boldsymbol{z}) \not\preceq \mathrm{pref}(\boldsymbol{d})$.

We consider the case $f'^{*}_{\langle\lambda\rangle}(\boldsymbol{z}) \preceq \mathrm{pref}(\boldsymbol{d})$ dividing into four cases by whether $\mathrm{pref}(\boldsymbol{d}) \prec f'^{*}_{\langle\lambda\rangle}(\boldsymbol{z})\boldsymbol{b}$ and whether $\mathrm{pref}(\boldsymbol{d}) \prec f'^{*}_{\langle\lambda\rangle}(\boldsymbol{z})\boldsymbol{b}'$.

- The case $\mathrm{pref}(\boldsymbol{d}) \prec f'^{*}_{\langle\lambda\rangle}(\boldsymbol{z})\boldsymbol{b}, \mathrm{pref}(\boldsymbol{d}) \prec f'^{*}_{\langle\lambda\rangle}(\boldsymbol{z})\boldsymbol{b}'$: We have

$$\psi_{\boldsymbol{z}}(\boldsymbol{b}) \overset{\text{(A)}}{=} f'^{*}_{\langle\lambda\rangle}(\boldsymbol{z})^{-1}\boldsymbol{d}\mathrm{pref}(\boldsymbol{d})^{-1}(f'^{*}_{\langle\lambda\rangle}(\boldsymbol{z})\boldsymbol{b}) \tag{3.289}$$

$$\overset{\text{(B)}}{\preceq} f'^{*}_{\langle\lambda\rangle}(\boldsymbol{z})^{-1}\boldsymbol{d}\mathrm{pref}(\boldsymbol{d})^{-1}(f'^{*}_{\langle\lambda\rangle}(\boldsymbol{z})\boldsymbol{b}') \tag{3.290}$$

$$\overset{\text{(C)}}{=} \psi_{\boldsymbol{z}}(\boldsymbol{b}') \tag{3.291}$$

as desired, where (A) follows from the first case of (3.58) and $\mathrm{pref}(\boldsymbol{d}) \prec f'^{*}_{\langle\lambda\rangle}(\boldsymbol{z})\boldsymbol{b}$, (B) follows from (3.287), and (C) follows from the first case of (3.58) and $\mathrm{pref}(\boldsymbol{d}) \prec f'^{*}_{\langle\lambda\rangle}(\boldsymbol{z})\boldsymbol{b}'$.

- The case $\mathrm{pref}(\boldsymbol{d}) \prec f'^{*}_{\langle\lambda\rangle}(\boldsymbol{z})\boldsymbol{b}, \mathrm{pref}(\boldsymbol{d}) \not\prec f'^{*}_{\langle\lambda\rangle}(\boldsymbol{z})\boldsymbol{b}'$: This case is impossible because (3.287) leads to $\mathrm{pref}(\boldsymbol{d}) \prec f'^{*}_{\langle\lambda\rangle}(\boldsymbol{z})\boldsymbol{b} \preceq f'^{*}_{\langle\lambda\rangle}(\boldsymbol{z})\boldsymbol{b}'$, which conflicts with $\mathrm{pref}(\boldsymbol{d}) \not\prec f'^{*}_{\langle\lambda\rangle}(\boldsymbol{z})\boldsymbol{b}'$.

- The case $\mathrm{pref}(\boldsymbol{d}) \not\prec f'^{*}_{\langle\lambda\rangle}(\boldsymbol{z})\boldsymbol{b}, \mathrm{pref}(\boldsymbol{d}) \prec f'^{*}_{\langle\lambda\rangle}(\boldsymbol{z})\boldsymbol{b}'$: By (3.287), we have

$$f'^{*}_{\langle\lambda\rangle}(\boldsymbol{z})\boldsymbol{b} \preceq f'^{*}_{\langle\lambda\rangle}(\boldsymbol{z})\boldsymbol{b}'. \tag{3.292}$$

By (3.292) and $\mathrm{pref}(\boldsymbol{d}) \prec f'^{*}_{\langle\lambda\rangle}(\boldsymbol{z})\boldsymbol{b}'$, exactly one of $\mathrm{pref}(\boldsymbol{d}) \prec f'^{*}_{\langle\lambda\rangle}(\boldsymbol{z})\boldsymbol{b}$ and $\mathrm{pref}(\boldsymbol{d}) \succeq f'^{*}_{\langle\lambda\rangle}(\boldsymbol{z})\boldsymbol{b}$ holds. Since the former does not hold by $\mathrm{pref}(\boldsymbol{d}) \not\prec f'^{*}_{\langle\lambda\rangle}(\boldsymbol{z})\boldsymbol{b}$, the latter holds:

$$f'^{*}_{\langle\lambda\rangle}(\boldsymbol{z})\boldsymbol{b} \preceq \mathrm{pref}(\boldsymbol{d}). \tag{3.293}$$

Thus, we have

$$\begin{aligned}
\psi_{\boldsymbol{z}}(\boldsymbol{b}) &\overset{\text{(A)}}{=} \boldsymbol{b} && (3.294)\\
&= f'^{*}_{\langle\lambda\rangle}(\boldsymbol{z})^{-1} f'^{*}_{\langle\lambda\rangle}(\boldsymbol{z})\boldsymbol{b} && (3.295)\\
&\overset{\text{(B)}}{\preceq} f'^{*}_{\langle\lambda\rangle}(\boldsymbol{z})^{-1}\mathrm{pref}(\boldsymbol{d}) && (3.296)\\
&\preceq f'^{*}_{\langle\lambda\rangle}(\boldsymbol{z})^{-1}\boldsymbol{d}\,\mathrm{pref}(\boldsymbol{d})^{-1}(f'^{*}_{\langle\lambda\rangle}(\boldsymbol{z})\boldsymbol{b}') && (3.297)\\
&\overset{\text{(C)}}{=} \psi_{\boldsymbol{z}}(\boldsymbol{b}'), && (3.298)
\end{aligned}$$

where (A) follows from the second case of (3.58) and $\mathrm{pref}(\boldsymbol{d}) \not\prec f'^{*}_{\langle\lambda\rangle}(\boldsymbol{z})\boldsymbol{b}$, (B) follows from (3.293), and (C) follows from the first case of (3.58) and $\mathrm{pref}(\boldsymbol{d}) \prec f'^{*}_{\langle\lambda\rangle}(\boldsymbol{z})\boldsymbol{b}'$.

- The case $\mathrm{pref}(\boldsymbol{d}) \not\prec f'^{*}_{\langle\lambda\rangle}(\boldsymbol{z})\boldsymbol{b}, \mathrm{pref}(\boldsymbol{d}) \not\prec f'^{*}_{\langle\lambda\rangle}(\boldsymbol{z})\boldsymbol{b}'$: We have

$$\psi_{\boldsymbol{z}}(\boldsymbol{b}) \overset{\text{(A)}}{=} \boldsymbol{b} \overset{\text{(B)}}{\preceq} \boldsymbol{b}' \overset{\text{(C)}}{=} \psi_{\boldsymbol{z}}(\boldsymbol{b}') \tag{3.299}$$

as desired, where (A) follows from the second case of (3.58) and $\mathrm{pref}(\boldsymbol{d}) \not\prec f'^{*}_{\langle\lambda\rangle}(\boldsymbol{z})\boldsymbol{b}$, (B) follows from (3.287), and (C) follows from the second case of (3.58) and $\mathrm{pref}(\boldsymbol{d}) \not\prec f'^{*}_{\langle\lambda\rangle}(\boldsymbol{z})\boldsymbol{b}'$.

(Proof of (ii)): We consider the following three cases separately: (I) the case $f'^{*}_{\langle\lambda\rangle}(\boldsymbol{z}) \preceq \mathrm{pref}(\boldsymbol{d}) \prec f'^{*}_{\langle\lambda\rangle}(\boldsymbol{zx})$, (II) the case $f'^{*}_{\langle\lambda\rangle}(\boldsymbol{zx}) \preceq \mathrm{pref}(\boldsymbol{d}) \prec f'^{*}_{\langle\lambda\rangle}(\boldsymbol{zx})\boldsymbol{c}$, and (III) the other case.

(I) The case $f'^*_{\langle\lambda\rangle}(\boldsymbol{z}) \preceq \mathrm{pref}(\boldsymbol{d}) \prec f'^*_{\langle\lambda\rangle}(\boldsymbol{zx})$: We have

$$f'^*_{\langle\lambda\rangle}(\boldsymbol{z}) \prec \boldsymbol{d} \preceq f'^*_{\langle\lambda\rangle}(\boldsymbol{zx}) \tag{3.300}$$

since

$$\mathrm{pref}(\boldsymbol{d})\bar{d}_l \overset{\text{(A)}}{\notin} \mathcal{P}^*_{F,\langle\lambda\rangle} \overset{\text{(B)}}{=} \mathcal{P}^*_{F',\langle\lambda\rangle}, \tag{3.301}$$

where (A) follows from (3.27), and (B) follows from Lemma 2.5.1 (ii) since φ defined in (3.30) is a homomorphism from F' to F. Therefore, by the second case of (3.58), we obtain

$$f''^*_{\langle\boldsymbol{z}\rangle}(\boldsymbol{x}) = f'^*_{\langle\lambda\rangle}(\boldsymbol{z})^{-1}\mathrm{pref}(\boldsymbol{d})\boldsymbol{d}^{-1}(f'^*_{\langle\lambda\rangle}(\boldsymbol{zx})). \tag{3.302}$$

We consider the following two cases separately: (I-A) the case $f'^*_{\langle\lambda\rangle}(\boldsymbol{z}) \prec f'^*_{\langle\lambda\rangle}(\boldsymbol{zx}) = \boldsymbol{d}, \boldsymbol{c} = \lambda$ and (I-B) the other case.

(I-A) The case $f'^*_{\langle\lambda\rangle}(\boldsymbol{z}) \prec f'^*_{\langle\lambda\rangle}(\boldsymbol{zx}) = \boldsymbol{d}, \boldsymbol{c} = \lambda$: We have

$$\begin{aligned}
f'^*_{\langle\lambda\rangle}(\boldsymbol{z})f''^*_{\langle\boldsymbol{z}\rangle}(\boldsymbol{x})\boldsymbol{c} &\overset{\text{(A)}}{=} f'^*_{\langle\lambda\rangle}(\boldsymbol{z})f'^*_{\langle\lambda\rangle}(\boldsymbol{z})^{-1}\mathrm{pref}(\boldsymbol{d})\boldsymbol{d}^{-1}(f'^*_{\langle\lambda\rangle}(\boldsymbol{zx}))\boldsymbol{c} && (3.303)\\
&\overset{\text{(B)}}{=} f'^*_{\langle\lambda\rangle}(\boldsymbol{z})f'^*_{\langle\lambda\rangle}(\boldsymbol{z})^{-1}\mathrm{pref}(\boldsymbol{d})\boldsymbol{d}^{-1}\boldsymbol{dc} && (3.304)\\
&\overset{\text{(C)}}{=} \mathrm{pref}(\boldsymbol{d}) && (3.305)\\
&\not\succ \mathrm{pref}(\boldsymbol{d}), && (3.306)
\end{aligned}$$

where (A) follows from (3.302), (B) follows from $f'^*_{\langle\lambda\rangle}(\boldsymbol{zx}) = \boldsymbol{d}$, and (C) follows from $\boldsymbol{c} = \lambda$.
Hence, we have

$$\begin{aligned}
\psi_{\boldsymbol{z}}(f''^*_{\langle\boldsymbol{z}\rangle}(\boldsymbol{x})\boldsymbol{c}) &\overset{\text{(A)}}{=} f''^*_{\langle\boldsymbol{z}\rangle}(\boldsymbol{x})\boldsymbol{c} && (3.307)\\
&\overset{\text{(B)}}{=} f'^*_{\langle\lambda\rangle}(\boldsymbol{z})^{-1}\mathrm{pref}(\boldsymbol{d})\boldsymbol{d}^{-1}f'^*_{\langle\lambda\rangle}(\boldsymbol{zx})\boldsymbol{c} && (3.308)\\
&\overset{\text{(C)}}{=} f'^*_{\langle\lambda\rangle}(\boldsymbol{z})^{-1}\mathrm{pref}(f'^*_{\langle\lambda\rangle}(\boldsymbol{zx}))\boldsymbol{d}^{-1}\boldsymbol{d} && (3.309)\\
&\overset{\text{(D)}}{=} f'^*_{\langle\lambda\rangle}(\boldsymbol{z})^{-1}f'^*_{\langle\lambda\rangle}(\boldsymbol{z})\mathrm{pref}(f'^*_{\langle\boldsymbol{z}\rangle}(\boldsymbol{x})))\boldsymbol{d}^{-1}\boldsymbol{d} && (3.310)\\
&= \mathrm{pref}(f'^*_{\langle\boldsymbol{z}\rangle}(\boldsymbol{x})) && (3.311)
\end{aligned}$$

as desired, where (A) follows from the second case of (3.58) and (3.306), (B) follows from (3.302), (C) follows from $f'^*_{\langle\lambda\rangle}(\boldsymbol{zx}) = \boldsymbol{d}$ and $\boldsymbol{c} = \lambda$, and (D) follows from Lemma 2.1.1 (i), Lemma 3.3.1 (i), and $f'^*_{\langle\lambda\rangle}(\boldsymbol{z}) \prec f'^*_{\langle\lambda\rangle}(\boldsymbol{zx})$.

(I-B) The other case: Then by (3.300), we have

$$\boldsymbol{d} \prec f'^{*}_{\langle\lambda\rangle}(\boldsymbol{zx})\boldsymbol{c}, \tag{3.312}$$

since it does not hold that $f'^{*}_{\langle\lambda\rangle}(\boldsymbol{z}) \prec f'^{*}_{\langle\lambda\rangle}(\boldsymbol{zx}) = \boldsymbol{d}, \boldsymbol{c} = \lambda$ by the assumption of the case (I-B).

We have

$$\begin{aligned} f'^{*}_{\langle\lambda\rangle}(\boldsymbol{z})f''^{*}_{\langle\boldsymbol{z}\rangle}(\boldsymbol{x})\boldsymbol{c} &\stackrel{\text{(A)}}{=} f'^{*}_{\langle\lambda\rangle}(\boldsymbol{z})f'^{*}_{\langle\lambda\rangle}(\boldsymbol{z})^{-1}\mathrm{pref}(\boldsymbol{d})\boldsymbol{d}^{-1}(f'^{*}_{\langle\lambda\rangle}(\boldsymbol{zx}))\boldsymbol{c} && (3.313)\\ &\stackrel{\text{(B)}}{\succ} f'^{*}_{\langle\lambda\rangle}(\boldsymbol{z})f'^{*}_{\langle\lambda\rangle}(\boldsymbol{z})^{-1}\mathrm{pref}(\boldsymbol{d})\boldsymbol{d}^{-1}\boldsymbol{d} && (3.314)\\ &= \mathrm{pref}(\boldsymbol{d}) && (3.315)\\ &\stackrel{\text{(C)}}{\succeq} f'^{*}_{\langle\lambda\rangle}(\boldsymbol{z}) && (3.316) \end{aligned}$$

as desired, where (A) follows from (3.302), (B) follows from (3.312), and (C) follows from the assumption of the case (I).

Hence, we have

$$\begin{aligned} &\psi_{\boldsymbol{z}}(f''_{\langle\boldsymbol{z}\rangle}(\boldsymbol{x})\boldsymbol{c})\\ &\stackrel{\text{(A)}}{=} f'^{*}_{\langle\lambda\rangle}(\boldsymbol{z})^{-1}\boldsymbol{d}\mathrm{pref}(\boldsymbol{d})^{-1}(f'^{*}_{\langle\lambda\rangle}(\boldsymbol{z})f''^{*}_{\langle\boldsymbol{z}\rangle}(\boldsymbol{x})\boldsymbol{c}) && (3.317)\\ &\stackrel{\text{(B)}}{=} f'^{*}_{\langle\lambda\rangle}(\boldsymbol{z})^{-1}\boldsymbol{d}\mathrm{pref}(\boldsymbol{d})^{-1}(f'^{*}_{\langle\lambda\rangle}(\boldsymbol{z})f'^{*}_{\langle\lambda\rangle}(\boldsymbol{z})^{-1}\mathrm{pref}(\boldsymbol{d})\boldsymbol{d}^{-1}(f'^{*}_{\langle\lambda\rangle}(\boldsymbol{zx})\boldsymbol{c})) && (3.318)\\ &= f'^{*}_{\langle\boldsymbol{z}\rangle}(\boldsymbol{x})\boldsymbol{c} && (3.319)\\ &\stackrel{\text{(C)}}{=} f'^{*}_{\langle\boldsymbol{z}\rangle}(\boldsymbol{x})\psi_{\boldsymbol{zx}}(\boldsymbol{c}), && (3.320) \end{aligned}$$

where (A) follows from the first case of (3.58), (3.315), and (3.316), (B) follows from (3.302), and (C) follows from the second case of (3.58) and the assumption of the case (I).

(II) The case $f'^{*}_{\langle\lambda\rangle}(\boldsymbol{zx}) \preceq \mathrm{pref}(\boldsymbol{d}) \prec f'^{*}_{\langle\lambda\rangle}(\boldsymbol{zx})\boldsymbol{c}$: Then since $\boldsymbol{d} \not\preceq f'^{*}_{\langle\lambda\rangle}(\boldsymbol{zx})$, we have

$$f''^{*}_{\langle\boldsymbol{z}\rangle}(\boldsymbol{x}) = f'^{*}_{\langle\boldsymbol{z}\rangle}(\boldsymbol{x}) \tag{3.321}$$

applying the second case of (3.37). Therefore, we have

$$
\begin{aligned}
f'^{*}_{\langle\lambda\rangle}(\boldsymbol{z}) &\preceq f'^{*}_{\langle\lambda\rangle}(\boldsymbol{zx}) && (3.322)\\
&\overset{\text{(A)}}{\preceq} \mathrm{pref}(\boldsymbol{d}) && (3.323)\\
&\overset{\text{(A)}}{\prec} f'^{*}_{\langle\lambda\rangle}(\boldsymbol{zx})\boldsymbol{c} && (3.324)\\
&\overset{\text{(B)}}{=} f'^{*}_{\langle\lambda\rangle}(\boldsymbol{z})f'^{*}_{\langle\boldsymbol{z}\rangle}(\boldsymbol{x})\boldsymbol{c} && (3.325)\\
&\overset{\text{(C)}}{=} f'^{*}_{\langle\lambda\rangle}(\boldsymbol{z})f''^{*}_{\langle\boldsymbol{z}\rangle}(\boldsymbol{x})\boldsymbol{c}, && (3.326)
\end{aligned}
$$

where (A)s follow from the assumption of the case (II), (B) follows from Lemma 2.1.1 (i) and Lemma 3.3.1 (i), and (C) follows from (3.321).

Hence, we have

$$
\begin{aligned}
\psi_{\boldsymbol{z}}(f''^{*}_{\langle\boldsymbol{z}\rangle}(\boldsymbol{x})\boldsymbol{c}) &\overset{\text{(A)}}{=} f'^{*}_{\langle\lambda\rangle}(\boldsymbol{z})^{-1}\boldsymbol{d}\mathrm{pref}(\boldsymbol{d})^{-1}(f'^{*}_{\langle\lambda\rangle}(\boldsymbol{z})f''^{*}_{\langle\boldsymbol{z}\rangle}(\boldsymbol{x})\boldsymbol{c}) && (3.327)\\
&\overset{\text{(B)}}{=} f'^{*}_{\langle\lambda\rangle}(\boldsymbol{z})^{-1}\boldsymbol{d}\mathrm{pref}(\boldsymbol{d})^{-1}(f'^{*}_{\langle\lambda\rangle}(\boldsymbol{z})f'^{*}_{\langle\boldsymbol{z}\rangle}(\boldsymbol{x})\boldsymbol{c}) && (3.328)\\
&\overset{\text{(C)}}{=} f'^{*}_{\langle\lambda\rangle}(\boldsymbol{z})^{-1}\boldsymbol{d}\mathrm{pref}(\boldsymbol{d})^{-1}(f'^{*}_{\langle\lambda\rangle}(\boldsymbol{zx})\boldsymbol{c}) && (3.329)\\
&= f'^{*}_{\langle\boldsymbol{z}\rangle}(\boldsymbol{x})f'^{*}_{\langle\boldsymbol{z}\rangle}(\boldsymbol{x})^{-1}f'^{*}_{\langle\lambda\rangle}(\boldsymbol{z})^{-1}\boldsymbol{d}\mathrm{pref}(\boldsymbol{d})^{-1}(f'^{*}_{\langle\lambda\rangle}(\boldsymbol{zx})\boldsymbol{c}) && (3.330)\\
&\overset{\text{(D)}}{=} f'^{*}_{\langle\boldsymbol{z}\rangle}(\boldsymbol{x})f'^{*}_{\langle\lambda\rangle}(\boldsymbol{zx})^{-1}\boldsymbol{d}\mathrm{pref}(\boldsymbol{d})^{-1}(f'^{*}_{\langle\lambda\rangle}(\boldsymbol{zx})\boldsymbol{c}) && (3.331)\\
&\overset{\text{(E)}}{=} f'^{*}_{\langle\boldsymbol{z}\rangle}(\boldsymbol{x})\psi_{\boldsymbol{zx}}(\boldsymbol{c}) && (3.332)
\end{aligned}
$$

as desired, where (A) follows from the first case of (3.58), (3.323), and (3.326), (B) follows from (3.321), (C) follows from Lemma 2.1.1 (i) and Lemma 3.3.1 (i), (D) follows from Lemma 2.1.1 (i) and Lemma 3.3.1 (i), and (E) follows from the first case of (3.58) and the assumption of the case (II).

(III) The other case: The following implication holds:

$$
f'^{*}_{\langle\lambda\rangle}(\boldsymbol{z}) \prec \boldsymbol{d} \preceq f'^{*}_{\langle\lambda\rangle}(\boldsymbol{zx}) \implies f'^{*}_{\langle\lambda\rangle}(\boldsymbol{z}) \preceq \mathrm{pref}(\boldsymbol{d}) \prec f'^{*}_{\langle\lambda\rangle}(\boldsymbol{zx}) \qquad (3.333)
$$

Now, it does not hold that $f'^{*}_{\langle\lambda\rangle}(\boldsymbol{z}) \preceq \mathrm{pref}(\boldsymbol{d}) \prec f'^{*}_{\langle\lambda\rangle}(\boldsymbol{zx})$ by the assumption of the case (III). Hence, by the contraposition of (3.333), we see that $f'^{*}_{\langle\lambda\rangle}(\boldsymbol{z}) \prec \boldsymbol{d} \preceq f'^{*}_{\langle\lambda\rangle}(\boldsymbol{zx})$ does not hold. Therefore, we obtain

$$
f''^{*}_{\langle\boldsymbol{z}\rangle}(\boldsymbol{x}) = f'^{*}_{\langle\boldsymbol{z}\rangle}(\boldsymbol{x}) \qquad (3.334)
$$

applying the second case of (3.37).

By the assumption of the case (III), neither $f'^*_{\langle\lambda\rangle}(\boldsymbol{z}) \preceq \mathrm{pref}(\boldsymbol{d}) \prec f'^*_{\langle\lambda\rangle}(\boldsymbol{zx})$ nor $f'^*_{\langle\lambda\rangle}(\boldsymbol{zx}) \preceq \mathrm{pref}(\boldsymbol{d}) \prec f'^*_{\langle\lambda\rangle}(\boldsymbol{zx})\boldsymbol{c}$ hold. Hence, the following condition does not hold:

$$f'^*_{\langle\lambda\rangle}(\boldsymbol{z}) \preceq \mathrm{pref}(\boldsymbol{d}) \prec f'^*_{\langle\lambda\rangle}(\boldsymbol{zx})\boldsymbol{c} \overset{\text{(A)}}{=} f'^*_{\langle\lambda\rangle}(\boldsymbol{z})f''^*_{\langle\boldsymbol{z}\rangle}(\boldsymbol{x})\boldsymbol{c}, \tag{3.335}$$

where (A) follows from (3.334). Therefore, by the second case of (3.58), we have

$$\psi_{\boldsymbol{z}}(f''^*_{\langle\boldsymbol{z}\rangle}(\boldsymbol{x})\boldsymbol{c}) = f''^*_{\langle\boldsymbol{z}\rangle}(\boldsymbol{x})\boldsymbol{c}. \tag{3.336}$$

Thus, we have

$$f'^*_{\langle\boldsymbol{z}\rangle}(\boldsymbol{x})\psi_{\boldsymbol{zx}}(\boldsymbol{c}) \overset{\text{(A)}}{=} f''^*_{\langle\boldsymbol{z}\rangle}(\boldsymbol{x})\psi_{\boldsymbol{zx}}(\boldsymbol{c}) \overset{\text{(B)}}{=} f''^*_{\langle\boldsymbol{z}\rangle}(\boldsymbol{x})\boldsymbol{c} \overset{\text{(C)}}{=} \psi_{\boldsymbol{z}}(f''^*_{\langle\boldsymbol{z}\rangle}(\boldsymbol{x})\boldsymbol{c}) \tag{3.337}$$

as desired, where (A) follows from (3.334), (B) follows from the second case of (3.58) since $f'^*_{\langle\lambda\rangle}(\boldsymbol{zx}) \preceq \mathrm{pref}(\boldsymbol{d}) \prec f'^*_{\langle\lambda\rangle}(\boldsymbol{zx})\boldsymbol{c}$ does not hold by the assumption of the case (III), and (C) follows from (3.336).

(Proof of (iii)): We have $f'^*_{\langle\lambda\rangle}(\boldsymbol{z}) \not\preceq \mathrm{pref}(\boldsymbol{d})$ because $|f'^*_{\langle\lambda\rangle}(\boldsymbol{z})| > |\boldsymbol{d}|$ by Lemma 3.3.2 (iii). Hence, by the second case of (3.58), we obtain $\psi_{\langle\boldsymbol{z}\rangle}(\boldsymbol{b}) = \boldsymbol{b}$ as desired. □

3.5.9 Proof of Lemma 3.4.5

To state the proof of Lemma 3.4.5, first we prove the following Lemma 3.5.1.

Lemma 3.5.1. *For $F(f,\tau) \in \mathscr{F}, i \in [F]$, and $\boldsymbol{x}, \boldsymbol{x}' \in \mathcal{S}^*$, if $\boldsymbol{x}' \preceq \boldsymbol{x}$ and $f^*_i(\boldsymbol{x}) = f^*_i(\boldsymbol{x}')$, then $\mathcal{P}^1_{F,\tau^*_i(\boldsymbol{x}')} \supseteq \mathcal{P}^1_{F,\tau^*_i(\boldsymbol{x})}$.*

Proof of Lemma 3.5.1. Choose $c \in \mathcal{P}^1_{F,\tau^*_i(\boldsymbol{x})}$ arbitrarily. Then there exists $\boldsymbol{y} \in \mathcal{S}^*$ such that

$$f^*_{\tau^*_i(\boldsymbol{x})}(\boldsymbol{y}) \succeq c. \tag{3.338}$$

We have

$$f^*_i(\boldsymbol{x}')c \overset{\text{(A)}}{=} f^*_i(\boldsymbol{x})c \tag{3.339}$$

$$\overset{\text{(B)}}{\preceq} f^*_i(\boldsymbol{x})f^*_{\tau^*_i(\boldsymbol{x})}(\boldsymbol{y}) \tag{3.340}$$

$$= f^*_i(\boldsymbol{x}'\boldsymbol{x}'^{-1}\boldsymbol{x})f^*_{\tau^*_i(\boldsymbol{x}'\boldsymbol{x}'^{-1}\boldsymbol{x})}(\boldsymbol{y}) \tag{3.341}$$

$$\overset{\text{(C)}}{=} f^*_i(\boldsymbol{x}')f^*_{\tau^*_i(\boldsymbol{x}')}(\boldsymbol{x}'^{-1}\boldsymbol{xy}), \tag{3.342}$$

where (A) follows from the assumption, (B) follows from (3.338), and (C) follows from Lemma 2.1.1. This yields $c \preceq f^*_{\tau^*_i(\boldsymbol{x}')}(\boldsymbol{x}'^{-1}\boldsymbol{x}\boldsymbol{y})$, which implies $c \in \mathcal{P}^1_{F,\tau^*_i(\boldsymbol{x}')}$. □

Proof of Lemma 3.4.5. Let $(k, i, \boldsymbol{x}, \boldsymbol{x}')$ be a tuple satisfying all of the conditions (a)–(c), and we lead a contradiction.

By the condition (c) and Lemma 2.1.1 (iii), we have

$$f^*_i(\boldsymbol{x}') \preceq f^*_i(\boldsymbol{x}). \tag{3.343}$$

Also, we have

$$|f^*_i(\boldsymbol{x})d_{F,\tau^*_i(\boldsymbol{x})}| + k \overset{\text{(A)}}{=} |d_{F,i}\widehat{f}^*_i(\boldsymbol{x})| + k \overset{\text{(B)}}{\leq} |d_{F,i}\widehat{f}^*_i(\boldsymbol{x}')| \overset{\text{(C)}}{=} |f^*_i(\boldsymbol{x}')d_{F,\tau^*_i(\boldsymbol{x}')}|, \tag{3.344}$$

where (A) follows from Lemma 3.4.1, (B) follows from the condition (b), (C) follows from Lemma 3.4.1. Hence, we have

$$|f^*_i(\boldsymbol{x}')| \geq |f^*_i(\boldsymbol{x})| + |d_{F,\tau^*_i(\boldsymbol{x})}| + k - |d_{F,\tau^*_i(\boldsymbol{x}')}|. \tag{3.345}$$

This yields

$$-1 \overset{\text{(A)}}{\leq} |d_{F,\tau^*_i(\boldsymbol{x})}| + k - |d_{F,\tau^*_i(\boldsymbol{x}')}| \overset{\text{(B)}}{\leq} 0, \tag{3.346}$$

where (A) follows since $0 \leq |d_{F,\tau^*_i(\boldsymbol{x})}| \leq 1$ and $0 \leq |d_{F,\tau^*_i(\boldsymbol{x}')}| \leq 1$, and (B) follows from (3.343) and (3.345). Therefore, the following two cases are possible: the case $|d_{F,\tau^*(\boldsymbol{x})}| + k - |d_{F,\tau^*(\boldsymbol{x}')}| = 0$ and the case $|d_{F,\tau^*(\boldsymbol{x})}| + k - |d_{F,\tau^*(\boldsymbol{x}')}| = -1$.

- The case $|d_{F,\tau^*_i(\boldsymbol{x})}| + k - |d_{F,\tau^*_i(\boldsymbol{x}')}| = 0$: Then by (3.345) and (3.343), we obtain

$$f^*_i(\boldsymbol{x}) = f^*_i(\boldsymbol{x}'). \tag{3.347}$$

  Also, the assumption that $|d_{F,\tau^*_i(\boldsymbol{x})}| + k - |d_{F,\tau^*_i(\boldsymbol{x}')}| = 0$ implies that the tuple $(k, |d_{F,\tau^*_i(\boldsymbol{x})}|, |d_{F,\tau^*_i(\boldsymbol{x}')}|)$ is equal to one of $(0, 0, 0), (0, 1, -1), (1, 0, -1)$. Now, the last case $(k, |d_{F,\tau^*_i(\boldsymbol{x})}|, |d_{F,\tau^*_i(\boldsymbol{x}')}|) = (1, 0, -1)$ is impossible because

$$\begin{aligned} |d_{F,\tau^*_i(\boldsymbol{x})}| = 0 \quad &\overset{\text{(A)}}{\Longleftrightarrow} \quad \mathcal{P}^1_{F,\tau^*_i(\boldsymbol{x})} = \{0, 1\} & (3.348)\\ &\overset{\text{(B)}}{\Longrightarrow} \quad \mathcal{P}^1_{F,\tau^*_i(\boldsymbol{x}')} = \{0, 1\} & (3.349)\\ &\overset{\text{(C)}}{\Longleftrightarrow} \quad |d_{F,\tau^*_i(\boldsymbol{x}')}| = 0, & (3.350) \end{aligned}$$

where (A) follows from (3.123), (B) follows since $\mathcal{P}^1_{F,\tau_i^*(\boldsymbol{x}')} \supseteq \mathcal{P}^1_{F,\tau_i^*(\boldsymbol{x})}$ by Lemma 3.5.1, and (C) follows from (3.123). Hence, we must have $k = 0$ and thus obtain $F \in \mathscr{F}_{0\text{-dec}}$ by the condition (a). Therefore, by (3.347) and Lemma 2.2.6 (ii), we obtain $\boldsymbol{x} = \boldsymbol{x}'$, which conflicts with the condition (b).

- The case $|d_{F,\tau_i^*(\boldsymbol{x})}| + k - |d_{F,\tau_i^*(\boldsymbol{x}')}| = -1$: Then we must have $k = |d_{F,\tau_i^*(\boldsymbol{x})}| = 0$ and $|d_{F,\tau_i^*(\boldsymbol{x}')}| = 1$ since $|d_{F,\tau_i^*(\boldsymbol{x})}| \geq 0$ and $|d_{F,\tau_i^*(\boldsymbol{x}')}| \leq 1$. Hence, we have
$$F \in \mathscr{F}_{0\text{-dec}} \tag{3.351}$$
by $k = 0$ and the condition (a), and we have
$$\mathcal{P}^1_{F,\tau_i^*(\boldsymbol{x}')} = \{d_{F,\tau_i^*(\boldsymbol{x}')}\} \tag{3.352}$$
by $|d_{F,\tau_i^*(\boldsymbol{x}')}| = 1$ and (3.123).
Also, we have
$$|f_i^*(\boldsymbol{x}')| \overset{\text{(A)}}{\leq} |f_i^*(\boldsymbol{x})| \overset{\text{(B)}}{\leq} |f_i^*(\boldsymbol{x}')| + 1, \tag{3.353}$$
where (A) follows from (3.343), and (B) follows from (3.345) and $|d_{F,\tau_i^*(\boldsymbol{x})}| + k - |d_{F,\tau_i^*(\boldsymbol{x}')}| = -1$. Therefore, we have either $|f_i^*(\boldsymbol{x})| = |f_i^*(\boldsymbol{x}')|$ or $|f_i^*(\boldsymbol{x})| + 1 = |f_i^*(\boldsymbol{x}')|$. If we assume $|f_i^*(\boldsymbol{x})| = |f_i^*(\boldsymbol{x}')|$, then $f_i^*(\boldsymbol{x}) = f_i^*(\boldsymbol{x}')$ holds by (3.343). Then by (3.351) and Lemma 2.2.6 (ii), we obtain $\boldsymbol{x} = \boldsymbol{x}'$, which conflicts with the condition (b). Hence, we have
$$|f_i^*(\boldsymbol{x})| = |f_i^*(\boldsymbol{x}')| + 1. \tag{3.354}$$
By the condition (b), there exists $\boldsymbol{z} = z_1 z_2 \ldots z_n \in \mathcal{S}^+$ such that $\boldsymbol{x} = \boldsymbol{x}'\boldsymbol{z}$. For such $\boldsymbol{z}$, we have
$$|f_i^*(\boldsymbol{x}')| + |f^*_{\tau_i^*(\boldsymbol{x}')}(\boldsymbol{z})| \overset{\text{(A)}}{=} |f_i^*(\boldsymbol{x}'\boldsymbol{z})| = |f_i^*(\boldsymbol{x})| \overset{\text{(B)}}{=} |f_i^*(\boldsymbol{x}')| + 1, \tag{3.355}$$
where (A) follows from Lemma 2.1.1 (i), and (B) follows from (3.354).
Choose $s \in \mathcal{S} \setminus \{z_n\}$ and define $\boldsymbol{z}' := \mathrm{suff}(\boldsymbol{z})s$. By $F \in \mathscr{F}_{\mathrm{ext}}$ and Lemma 2.3.1, we can choose $\boldsymbol{y} \in \mathcal{S}^*$ such that
$$|f^*_{\tau_i^*(\boldsymbol{x}')}(\boldsymbol{z}'\boldsymbol{y})| \geq 1. \tag{3.356}$$
Then by (3.352), we have
$$f^*_{\tau_i^*(\boldsymbol{x}')}(\boldsymbol{z}) \overset{\text{(A)}}{=} d_{F,\tau_i^*(\boldsymbol{x}')} \overset{\text{(B)}}{\preceq} f^*_{\tau_i^*(\boldsymbol{x}')}(\boldsymbol{z}'\boldsymbol{y}), \tag{3.357}$$

where (A) follows since $|f^*_{\tau^*_i(\boldsymbol{x}')}(\boldsymbol{z})| = 1$ by (3.355), and (B) follows from (3.356). By (3.351), (3.357) and Lemma 2.2.6 (i), we obtain $\boldsymbol{z} \preceq \boldsymbol{z}'\boldsymbol{y}$. This conflicts with the definition of $\boldsymbol{z}'$.

$\square$

Chapter 4

Optimality of Huffman Codes and AIFV Codes

In this chapter, by applying the three theorems in the previous chapter, we prove that the class of Huffman codes (resp. AIFV codes) achieves the optimal average codeword length in $\mathscr{F}_{\text{reg}} \cap \mathscr{F}_{\text{ext}} \cap \mathscr{F}_{1\text{-dec}}$ (resp. $\mathscr{F}_{\text{reg}} \cap \mathscr{F}_{\text{ext}} \cap \mathscr{F}_{2\text{-dec}}$) in Section 4.1 (resp. Section 4.2). Note that we are now discussing an arbitrarily fixed source distribution μ.

4.1 Optimality of Huffman Codes in the Class of 1-bit Delay Decodable Codes

The main result of this subsection is the following Theorem 4.1.1 that the Huffman code achieves the optimal average codeword length in $\mathscr{F}_{\text{reg}} \cap \mathscr{F}_{\text{ext}} \cap \mathscr{F}_{1\text{-dec}}$.

Theorem 4.1.1. *For any $F \in \mathscr{F}_{\text{reg}} \cap \mathscr{F}_{\text{ext}} \cap \mathscr{F}_{1\text{-dec}}$, we have*

$$L(F) \geq L_{\text{Huff}}, \tag{4.1}$$

where L_{Huff} is the average codeword length of the Huffman code.

The proof of Theorem 4.1.1 relies on the following Lemma 4.1.1.

Lemma 4.1.1. $\mathscr{F}_{\text{reg}} \cap \mathscr{F}_{\text{ext}} \cap \mathscr{F}_{1\text{-dec}} \cap \mathscr{F}_{\text{fork}} \subseteq \mathscr{F}_{0\text{-dec}}$.

Proof of Lemma 4.1.1. Choose $F(f, \tau) \in \mathscr{F}_{\text{reg}} \cap \mathscr{F}_{\text{ext}} \cap \mathscr{F}_{\text{1-dec}} \cap \mathscr{F}_{\text{fork}}$ arbitrarily. We prove $F \in \mathscr{F}_{\text{0-dec}}$ by showing that for $i \in [F]$ and $\boldsymbol{x} \in \mathcal{S}^*$, the pair $(\boldsymbol{x}, \lambda)$ is f_i^*-positive, that is, for any $i \in [F]$ and $\boldsymbol{x}, \boldsymbol{x'} \in \mathcal{S}^*$ such that $f_i^*(\boldsymbol{x}) \preceq f_i^*(\boldsymbol{x'})$, we have $\boldsymbol{x} \preceq \boldsymbol{x'}$.

Choose $\boldsymbol{x}, \boldsymbol{x'} \in \mathcal{S}^*$ such that $f_i^*(\boldsymbol{x}) \preceq f_i^*(\boldsymbol{x'})$ arbitrarily. Since $F \in \mathscr{F}_{\text{fork}}$, we have $\mathcal{P}^1_{F,\tau_i^*(\boldsymbol{x})} = \{0, 1\}$, that is, there exist $\boldsymbol{y}_0, \boldsymbol{y}_1 \in \mathcal{S}^*$ such that $f^*_{\tau_i^*(\boldsymbol{x})}(\boldsymbol{y}_0) \succeq 0$ and $f^*_{\tau_i^*(\boldsymbol{x})}(\boldsymbol{y}_1) \succeq 1$. Hence, we have

$$f_i^*(\boldsymbol{x}\boldsymbol{y}_0) \overset{\text{(A)}}{=} f_i^*(\boldsymbol{x}) f^*_{\tau_i^*(\boldsymbol{x})}(\boldsymbol{y}_0) \succeq f_i^*(\boldsymbol{x})0, \tag{4.2}$$

$$f_i^*(\boldsymbol{x}\boldsymbol{y}_1) \overset{\text{(B)}}{=} f_i^*(\boldsymbol{x}) f^*_{\tau_i^*(\boldsymbol{x})}(\boldsymbol{y}_1) \succeq f_i^*(\boldsymbol{x})1, \tag{4.3}$$

where (A) and (B) follow from Lemma 2.1.1 (i). Since $\boldsymbol{x} \preceq \boldsymbol{x}\boldsymbol{y}_0, \boldsymbol{x} \preceq \boldsymbol{x}\boldsymbol{y}_1$, and $F \in \mathscr{F}_{\text{1-dec}}$, the pairs $(\boldsymbol{x}, 0), (\boldsymbol{x}, 1)$ are f_i^*-positive.

By $f_i^*(\boldsymbol{x}) \preceq f_i^*(\boldsymbol{x'})$ and $F \in \mathscr{F}_{\text{ext}}$, there exist $c, c' \in \mathcal{C}$ and $\boldsymbol{x''} \in \mathcal{S}^*$ such that $f_i^*(\boldsymbol{x})c \preceq f_i^*(\boldsymbol{x'})c' \preceq f_i^*(\boldsymbol{x'}\boldsymbol{x''})$. Since $(\boldsymbol{x}, 0)$ and $(\boldsymbol{x}, 1)$ are $f_i'^*$-positive, we have $\boldsymbol{x} \preceq \boldsymbol{x'}\boldsymbol{x''}$. Therefore, we have either (a) or (b) of the following conditions: (a) $\boldsymbol{x} \preceq \boldsymbol{x'}$; (b) $\boldsymbol{x} \succ \boldsymbol{x'}$. To complete the proof, it suffices to prove that (a) is true. Now we prove it by contradiction assuming that (b) is true, that is, there exists $\boldsymbol{z} = z_1 z_2 \ldots z_n \in \mathcal{S}^+$ such that $\boldsymbol{x} = \boldsymbol{x'}\boldsymbol{z}$.

By $\boldsymbol{x} \succ \boldsymbol{x'}$, Lemma 2.1.1 (iii), and $f_i^*(\boldsymbol{x}) \preceq f_i^*(\boldsymbol{x'})$, we have

$$f_i^*(\boldsymbol{x'}) = f_i^*(\boldsymbol{x}). \tag{4.4}$$

Choose $s \in \mathcal{S} \setminus \{z_n\}$ and define $\boldsymbol{z'} = \text{pref}(\boldsymbol{z})s$. By $F' \in \mathscr{F}_{\text{ext}}$, we can choose $\boldsymbol{y'} \in \mathcal{S}^*$ and $c' \in \mathcal{C}$ such that

$$f^*_{\tau'^*_i(\boldsymbol{x'})}(\boldsymbol{z'}\boldsymbol{y'}) \succeq c'. \tag{4.5}$$

By (4.4) and (4.5), we have

$$f_i^*(\boldsymbol{x})c = f_i^*(\boldsymbol{x'})c' \preceq f_i^*(\boldsymbol{x'}) f^*_{\tau'^*_i(\boldsymbol{x'})}(\boldsymbol{z'}\boldsymbol{y'}) \overset{\text{(A)}}{=} f_i^*(\boldsymbol{x'}\boldsymbol{z'}\boldsymbol{y'}), \tag{4.6}$$

where (A) follows from Lemma 2.1.1 (i). Since the pairs $(\boldsymbol{x}, 0)$ and $(\boldsymbol{x}, 1)$ are f_i^*-positive (in particular, $(\boldsymbol{x}, c')$ is f_i^*-positive), we have $\boldsymbol{x} \preceq \boldsymbol{x'}\boldsymbol{z'}\boldsymbol{y'}$. By $\boldsymbol{x} = \boldsymbol{x'}\boldsymbol{z}$, we have $\boldsymbol{x'}\boldsymbol{z} \preceq \boldsymbol{x'}\boldsymbol{z'}\boldsymbol{y'}$. Hence, we obtain $\boldsymbol{z} \preceq \boldsymbol{z'}\boldsymbol{y'}$. This conflicts with the definition of $\boldsymbol{z'}$. □

Proof of Theorem 4.1.1. Choose $F \in \mathscr{F}_{\text{reg}} \cap \mathscr{F}_{\text{ext}} \cap \mathscr{F}_{\text{1-dec}}$ arbitrarily. By Theorem 3.1.3, there exists $F' \in \mathscr{F}_{\text{reg}} \cap \mathscr{F}_{\text{ext}} \cap \mathscr{F}_{\text{1-dec}} \cap \mathscr{F}_{\text{fork}}$ such that $L(F') = L(F)$. By Theorem 3.1.1, there exists $F^\dagger \in \mathscr{F}_{\text{irr}} \cap \mathscr{F}_{\text{ext}} \cap \mathscr{F}_{\text{0-dec}}$ such that $L(F^\dagger) \leq L(F')$ and

$$|F^\dagger| \overset{\text{(A)}}{=} |\mathscr{P}^1_{F^\dagger}| \overset{\text{(B)}}{\leq} |\mathscr{P}^1_{F'}| \overset{\text{(C)}}{=} |\{\{0,1\}\}| = 1, \tag{4.7}$$

where (A) follows from Theorem 3.1.1 (d), (B) follows from Theorem 3.1.1 (c), and (C) follows from $F' \in \mathscr{F}_{\text{fork}}$. By Lemma 4.1.1, we have $F' \in \mathscr{F}_{\text{irr}} \cap \mathscr{F}_{\text{ext}} \cap \mathscr{F}_{\text{1-dec}} \subseteq \mathscr{F}_{\text{0-dec}}$. Hence, by Lemma 2.2.6, the only code table $f^\dagger_0$ of $F^\dagger$ is injective, and thus $F^\dagger$ is a uniquely decodable code with a single code table. Therefore, by McMillan's Theorem [2], we have $L(F^\dagger) \geq L_{\text{Huff}}$ so that

$$L(F) = L(F') \geq L(F^\dagger) \geq L_{\text{Huff}} \tag{4.8}$$

as desired. □

4.2 Optimality of AIFV Codes in the Class of 2-bit Delay Decodable Codes

In this section, we prove that the class of AIFV codes achieves the optimal average codeword length in $\mathscr{F}_{\text{reg}} \cap \mathscr{F}_{\text{ext}} \cap \mathscr{F}_{\text{2-dec}}$. The class of AIFV codes $\mathscr{F}_{\text{AIFV}}$ is formalized with our notations as the following Definition 4.2.1.

Definition 4.2.1. *We define $\mathscr{F}_{\text{AIFV}}$ as the set of all $F(f, \tau) \in \mathscr{F}^{(2)}$ satisfying all of the following conditions (i)–(vii).*

(i) f_0 and f_1 are injective.

(ii) For any $i \in [2]$ and $s \in \mathcal{S}$, it holds that $\bar{\mathcal{P}}^1_{F,i}(f_i(s)) \not\ni 1$ and $\bar{\mathcal{P}}^1_{F,i}(f_i(s)0) \not\ni 1$.

(iii) For any $i \in [2]$ and $s, s' \in \mathcal{S}$, it holds that $f_i(s') \neq f_i(s)0$.

(iv) For any $i \in [2]$ and $s \in \mathcal{S}$, it holds that

$$\tau_i(s) = \begin{cases} 0 & \text{if } \bar{\mathcal{P}}^0_{F,i}(f_i(s)) = \emptyset, \\ 1 & \text{if } \bar{\mathcal{P}}^0_{F,i}(f_i(s)) \neq \emptyset. \end{cases} \tag{4.9}$$

(v) *For any* $s \in \mathcal{S}$*, it holds that* $f_1(s) \neq \lambda$ *and* $f_1(s) \neq 0$*.*

(vi) $\bar{\mathcal{P}}^1_{F,1}(0) \not\ni 0$*.*

(vii) *For any* $i \in [2]$ *and* $\boldsymbol{b} \in \mathcal{C}^*$*, if* $|\bar{\mathcal{P}}^1_{F,i}(\boldsymbol{b})| = 1$*, then at least one of the following conditions (a) and (b) hold.*

(a) $f_i(s)\boldsymbol{c} = \boldsymbol{b}$ *for some* $s \in \mathcal{S}$ *and* $\boldsymbol{c} \in \mathcal{C}^0 \cup \mathcal{C}^1$*.*

(b) $(i, \boldsymbol{b}) = (1, 0)$*.*

Example 4.2.1. *The code-tuple* $F^{(\kappa)}$ *in Table 4.1 is in* $\mathscr{F}_{\mathrm{AIFV}}$*.*

Now, the desired theorem, the optimality of AIFV codes in $\mathscr{F}_{\mathrm{reg}} \cap \mathscr{F}_{\mathrm{ext}} \cap \mathscr{F}_{\text{2-dec}}$, can be stated as follows.

Theorem 4.2.1. $\mathscr{F}_{\text{2-opt}} \cap \mathscr{F}_{\mathrm{AIFV}} \neq \emptyset$*.*

Theorem 4.2.1 claims that there exists an optimal AIFV code, that is, the class of AIFV codes achieves the optimal average codeword length in $\mathscr{F}_{\mathrm{reg}} \cap \mathscr{F}_{\mathrm{ext}} \cap \mathscr{F}_{\text{2-dec}}$. We prove Theorem 4.2.1 through this section. To prove this, we introduce four classes of code-tuples $\mathscr{F}_0$, $\mathscr{F}_1$, $\mathscr{F}_2$, $\mathscr{F}_3$ and $\mathscr{F}_4$, as follows.

Definition 4.2.2. *We define* $\mathscr{F}_0, \mathscr{F}_1$*,* $\mathscr{F}_2$*,* $\mathscr{F}_3$ *and* $\mathscr{F}_4$ *as follows.*

- $\mathscr{F}_0 = \mathscr{F}_{\mathrm{reg}} \cap \mathscr{F}_{\mathrm{ext}} \cap \mathscr{F}_{\text{2-dec}} = \{F \in \mathscr{F}_{\mathrm{reg}} \cap \mathscr{F}_{\text{2-dec}} : {}^\forall i \in [F]; \mathcal{P}^1_{F,i} \neq \emptyset\}$.
- $\mathscr{F}_1 = \mathscr{F}_{\mathrm{reg}} \cap \mathscr{F}_{\mathrm{ext}} \cap \mathscr{F}_{\text{2-dec}} \cap \mathscr{F}_{\mathrm{fork}} = \{F \in \mathscr{F}_{\mathrm{reg}} \cap \mathscr{F}_{\text{2-dec}} : {}^\forall i \in [F]; \mathcal{P}^1_{F,i} = \{0, 1\}\}$.
- $\mathscr{F}_2 = \{F \in \mathscr{F}_{\mathrm{reg}} \cap \mathscr{F}_{\text{2-dec}} : {}^\forall i \in [F]; |\mathcal{P}^2_{F,i}| \geq 3\}$.
- $\mathscr{F}_3 = \{F \in \mathscr{F}_{\mathrm{reg}} \cap \mathscr{F}_{\text{2-dec}} : {}^\forall i \in [F]; \mathcal{P}^2_{F,i} \supseteq \{01, 10, 11\}\}$.
- $\mathscr{F}_4 = \{F \in \mathscr{F}_{\mathrm{reg}} \cap \mathscr{F}_{\text{2-dec}} \cap \mathscr{F}^{(2)} : \mathcal{P}^2_{F,0} = \{00, 01, 10, 11\}, \mathcal{P}^2_{F,1} = \{01, 10, 11\}\}$.

By the definitions, the classes defined above form a hierarchical structure as follows:

$$\mathscr{F}_0 \supseteq \mathscr{F}_1 \overset{\text{(A)}}{\supseteq} \mathscr{F}_2 \supseteq \mathscr{F}_3 \supseteq \mathscr{F}_4 \overset{\text{(B)}}{\supseteq} \mathscr{F}_{\mathrm{AIFV}}, \tag{4.10}$$

where (A) follows from Lemma 2.3.2 (i), and (B) is stated as the following Lemma 4.2.1, which proof is in Subsection 4.3.1.

Table 4.1: Examples of a code-tuple $F^{(\gamma)}$–$F^{(\kappa)}$

$s \in \mathcal{S}$	$f_0^{(\gamma)}$	$\tau_0^{(\gamma)}$	$f_1^{(\gamma)}$	$\tau_1^{(\gamma)}$	$f_2^{(\gamma)}$	$\tau_2^{(\gamma)}$
a	01	0	00	1	1100	1
b	10	1	λ	0	1110	0
c	0100	0	00111	1	111000	2
d	01	2	00111	2	110	2

$s \in \mathcal{S}$	$f_0^{(\delta)}$	$\tau_0^{(\delta)}$	$f_1^{(\delta)}$	$\tau_1^{(\delta)}$	$f_2^{(\delta)}$	$\tau_2^{(\delta)}$
a	01	0	00	1	100	1
b	10	1	λ	0	110	0
c	0100	0	00111	1	110001	2
d	011	2	001111	2	101	2

$s \in \mathcal{S}$	$f_0^{(\epsilon)}$	$\tau_0^{(\epsilon)}$	$f_1^{(\epsilon)}$	$\tau_1^{(\epsilon)}$	$f_2^{(\epsilon)}$	$\tau_2^{(\epsilon)}$
a	01	0	00	1	00	1
b	10	1	λ	0	10	0
c	0100	0	00111	1	100011	2
d	0111	2	0011111	2	011	2

$s \in \mathcal{S}$	$f_0^{(\zeta)}$	$\tau_0^{(\zeta)}$	$f_1^{(\zeta)}$	$\tau_1^{(\zeta)}$	$f_2^{(\zeta)}$	$\tau_2^{(\zeta)}$
a	10	0	01	1	00	1
b	11	1	λ	0	10	0
c	1000	0	01001	1	100011	2
d	1001	2	0100100	2	011	2

$s \in \mathcal{S}$	$f_0^{(\eta)}$	$\tau_0^{(\eta)}$	$f_1^{(\eta)}$	$\tau_1^{(\eta)}$	$f_2^{(\eta)}$	$\tau_2^{(\eta)}$
a	01	0	01	1	00	1
b	1	1	1	0	101	0
c	0001	0	01001	1	100011	2
d	001	2	0100100	2	011	2

$s \in \mathcal{S}$	$f_0^{(\theta)}$	$\tau_0^{(\theta)}$	$f_1^{(\theta)}$	$\tau_1^{(\theta)}$	$f_2^{(\theta)}$	$\tau_2^{(\theta)}$
a	01	0	01	1	10	1
b	1	1	1	0	011	0
c	0001	0	01001	1	010011	2
d	001	2	0100100	2	111	2

$s \in \mathcal{S}$	$f_0^{(\iota)}$	$\tau_0^{(\iota)}$	$f_1^{(\iota)}$	$\tau_1^{(\iota)}$
a	01	1	01	1
b	1	1	1	0
c	0001	0	01001	1
d	001	1	0100100	1

$s \in \mathcal{S}$	$f_0^{(\kappa)}$	$\tau_0^{(\kappa)}$	$f_1^{(\kappa)}$	$\tau_1^{(\kappa)}$
a	100	0	1100	0
b	00	0	11	1
c	01	0	01	0
d	1	1	10	0

Table 4.2: The set $\mathcal{P}^2_{F,i}$ for the code-tuples F in Table 4.1

$F \in \mathscr{F}$	$\mathcal{P}^2_{F,0}$	$\mathcal{P}^2_{F,1}$	$\mathcal{P}^2_{F,2}$	
$F^{(\gamma)}$	$\{01, 10\}$	$\{00, 01, 10\}$	$\{11\}$	$F \in \mathscr{F}_0 \setminus \mathscr{F}_1$
$F^{(\delta)}$	$\{01, 10\}$	$\{00, 01, 10\}$	$\{10, 11\}$	$F \in \mathscr{F}_0 \setminus \mathscr{F}_1$
$F^{(\epsilon)}$	$\{01, 10\}$	$\{00, 01, 10\}$	$\{00, 01, 10\}$	$F \in \mathscr{F}_1 \setminus \mathscr{F}_2$
$F^{(\zeta)}$	$\{10, 11\}$	$\{01, 10, 11\}$	$\{00, 01, 10\}$	$F \in \mathscr{F}_0 \setminus \mathscr{F}_1$
$F^{(\eta)}$	$\{00, 01, 10, 11\}$	$\{01, 10, 11\}$	$\{00, 01, 10\}$	$F \in \mathscr{F}_2 \setminus \mathscr{F}_3$
$F^{(\theta)}$	$\{00, 01, 10, 11\}$	$\{01, 10, 11\}$	$\{01, 10, 11\}$	$F \in \mathscr{F}_3 \setminus \mathscr{F}_4$
$F^{(\iota)}$	$\{00, 01, 10, 11\}$	$\{01, 10, 11\}$		$F \in \mathscr{F}_4 \setminus \mathscr{F}_{\mathrm{AIFV}}$
$F^{(\kappa)}$	$\{00, 01, 10, 11\}$	$\{01, 10, 11\}$		$F \in \mathscr{F}_{\mathrm{AIFV}}$

Lemma 4.2.1. $\mathscr{F}_4 \supseteq \mathscr{F}_{\mathrm{AIFV}}$.

Example 4.2.2. *The rightmost column of Table 4.2 indicates the class to which each code-tuple in Table 4.1 belongs.*

Noting that $\mathscr{F}_{2\text{-opt}} = \arg\min_{F\in\mathscr{F}_0} L(F)$, we have $\mathscr{F}_{2\text{-opt}} \cap \mathscr{F}_0 \neq \emptyset$ directly from Definition 3.1.1 and Lemma 3.1.2. Starting from this, we sequentially prove $\mathscr{F}_{2\text{-opt}} \cap \mathscr{F}_i \neq \emptyset$ for $i = 1, 2, 3, 4$, in Subsection 4.2.1–4.2.4, respectively. Then in Subsection 4.2.5, we finally prove the desired Theorem 4.2.1 from $\mathscr{F}_{2\text{-opt}} \cap \mathscr{F}_4 \neq \emptyset$.

We use the following Lemma 4.2.2 throughout this section.

Lemma 4.2.2. *For any integer $k \leq 2$, $F(f, \tau) \in \mathscr{F}_{2\text{-dec}} \cap \mathscr{F}_{\mathrm{ext}}$, $i \in [F]$, and $s \in \mathcal{S}$, we have $|\bar{\mathcal{P}}^k_{F,i}(f_i(s))| + |\mathcal{P}^2_{F,\tau_i(s)}| \leq 4$.*

Proof of Lemma 4.2.2. We have

$$|\bar{\mathcal{P}}^k_{F,i}(f_i(s))| + |\mathcal{P}^2_{F,\tau_i(s)}| \overset{\text{(A)}}{\leq} |\bar{\mathcal{P}}^2_{F,i}(f_i(s))| + |\mathcal{P}^2_{F,\tau_i(s)}| \overset{\text{(B)}}{\leq} |\mathcal{P}^2_{F,i}(f_i(s))| \leq 4 \quad (4.11)$$

as desired, where (A) follows from $k \leq 2$, $F \in \mathscr{F}_{\text{ext}}$, and Corollary 2.3.1 (ii) (b), and (B) follows from $F \in \mathscr{F}_{\text{2-dec}}$ and Lemma 2.2.3. □

4.2.1 The Class $\mathscr{F}_1$

Since $\mathscr{F}_1 = \mathscr{F}_{\text{reg}} \cap \mathscr{F}_{\text{ext}} \cap \mathscr{F}_{\text{2-dec}} \cap \mathscr{F}_{\text{fork}}$, we obtain the following Lemma 4.2.3 immediately from Corollary 3.1.1.

Lemma 4.2.3. $\mathscr{F}_{\text{2-opt}} \cap \mathscr{F}_1 \neq \emptyset$.

We enumerate the basic properties of $\mathscr{F}_1$ as the following Lemmas 4.2.4 and 4.2.5. See subsections 4.3.2 and 4.3.3 for the proofs of Lemmas 4.2.4 and 4.2.5, respectively.

Lemma 4.2.4. *For any $F(f,\tau) \in \mathscr{F}_1$ and $i \in [F]$, the following statements (i)–(vi) hold.*

(i) $\mathcal{P}^2_{F,i} \supseteq \{0a, 1b\}$ *for some* $a, b \in \mathcal{C}$. *In particular,* $|\mathcal{P}^2_{F,i}| \geq 2$.

(ii) If $|\mathcal{P}^2_{F,i}| = 2$, *then the following statements (a) and (b) hold.*

  (a) For any $s \in \mathcal{S}$, *we have* $|f_i(s)| \geq 2$.

  (b) $\mathcal{P}^2_{F,i} = \bar{\mathcal{P}}^2_{F,i} = \{0a, 1b\}$ *for some* $a, b \in \mathcal{C}$.

(iii) For any $s, s' \in \mathcal{S}$, *if* $s \neq s'$ *and* $f_i(s) = f_i(s')$, *then* $|\mathcal{P}^2_{F,\tau_i(s)}| = |\mathcal{P}^2_{F,\tau_i(s')}| = 2$.

(iv) For any $s \in \mathcal{S}$, *we have*

$$|\mathcal{S}_{F,i}(f_i(s))| \leq \begin{cases} 1 & \text{if } \bar{\mathcal{P}}^0_{F,i}(f_i(s)) \neq \emptyset, \\ 2 & \text{if } \bar{\mathcal{P}}^0_{F,i}(f_i(s)) = \emptyset. \end{cases} \quad (4.12)$$

(v) For any $s, s' \in \mathcal{S}$, *we have* $f_i(s') \neq f_i(s)0$ *and* $f_i(s') \neq f_i(s)1$.

(vi) For any $s \in \mathcal{S}$, *we have* $|\bar{\mathcal{P}}^1_{F,i}(f_i(s)0)| \leq 1$ *and* $|\bar{\mathcal{P}}^1_{F,i}(f_i(s)1)| \leq 1$.

Lemma 4.2.5. *For any* $F(f,\tau) \in \mathscr{F}_{\text{2-opt}} \cap \mathscr{F}_1, i \in \mathcal{R}_F$ *and* $s \in \mathcal{S}$, *if* $\bar{\mathcal{P}}^0_{F,i}(f_i(s)) = \emptyset$ *and* $|\mathcal{S}_{F,i}(f_i(s))| = 1$, *then* $|\mathcal{P}^2_{F,\tau_i(s)}| = 4$.

4.2.2 The Class $\mathscr{F}_2$

In this subsection, we prove $\mathscr{F}_{2\text{-opt}} \cap \mathscr{F}_2 \neq \emptyset$ and some properties of the class $\mathscr{F}_2$.

- First, we define an operation called *dot operation*, which transforms a given code-tuple $F \in \mathscr{F}_1$ into the code-tuple $\dot{F}$ defined as Definition 4.2.4.
- Next, we consider the code-tuple $\widehat{\dot{F}}$, obtained from F by applying dot operation firstly and rotation secondly. We show that $\widehat{\dot{F}} \in \mathscr{F}_1$ and $L(\widehat{\dot{F}}) = L(F)$ hold for any $F \in \mathscr{F}_1$.
- Then we show that we can transform any $F \in \mathscr{F}_{2\text{-opt}} \cap \mathscr{F}_1$ into some $F' \in \mathscr{F}_{2\text{-opt}} \cap \mathscr{F}_2$ by repeating dot operation and rotation alternately. This shows $\mathscr{F}_{2\text{-opt}} \cap \mathscr{F}_2 \neq \emptyset$ since $\mathscr{F}_{2\text{-opt}} \cap \mathscr{F}_1 \neq \emptyset$ by Lemma 4.2.3.

To state the definition of $\dot{F}$, we first introduce the decomposition of a codeword called *γ-decomposition*. Fix $F(f, \tau) \in \mathscr{F}_1, i \in [F]$, and $s \in \mathcal{S}$, and define $\mathcal{S}^{\prec}_{F,i}(f_i(s)) \coloneqq \{s' \in \mathcal{S} : f_i(s') \prec f_i(s)\}$. By Lemma 2.2.2 (i), we have $|\bar{\mathcal{P}}^0_{F,i}(f_i(s'))| \neq \emptyset$ for any $s' \in \mathcal{S}^{\prec}_{F,i}(f_i(s))$, which leads to $|\mathcal{S}_{F,i}(f_i(s'))| = 1$ by Lemma 4.2.4 (iv). Thus, without loss of generality, we may assume

$$f_i(s_1) \prec f_i(s_2) \prec \cdots \prec f_i(s_\rho), \tag{4.13}$$

where $\mathcal{S}^{\prec}_{F,i}(f_i(s)) = \{s_1, s_2, \ldots, s_{\rho-1}\}$ and $s_\rho \coloneqq s$. Then there uniquely exist $\gamma(s_1), \gamma(s_2), \ldots, \gamma(s_\rho) \in \mathcal{C}^*$ such that

$$f_i(s_r) = \begin{cases} \gamma(s_1) & \text{if } r = 1, \\ f_i(s_{r-1})\gamma(s_r) & \text{if } r = 2, 3, \ldots, \rho \end{cases} \tag{4.14}$$

for any $r = 1, 2, \ldots, \rho$. We can represent $f_i(s)$ as

$$f_i(s) = \gamma(s_1)\gamma(s_2)\ldots\gamma(s_\rho). \tag{4.15}$$

Definition 4.2.3. *For $F(f, \tau) \in \mathscr{F}_1, i \in [F]$, and $s \in \mathcal{S}$, we define* γ-decomposition *of $f_i(s)$ as the representation in (4.15). Note that $s_\rho = s$.*

Example 4.2.3. *We consider $F(f, \tau) \coloneqq F^{(\epsilon)}$ in Table 4.1.*

- *First, we consider the* γ*-decomposition of* $f_1(\mathrm{d})$*. We have* $\mathcal{S}^{\prec}_{F,1}(f_1(\mathrm{d})) = \{\mathrm{a}, \mathrm{b}, \mathrm{c}\}$*. Since* $f_1(\mathrm{b}) = \lambda \prec f_1(\mathrm{a}) = 00 \prec f_1(\mathrm{c}) = 00111$*. Thus, we obtain the* γ*-decomposition of* $f_1(\mathrm{d})$ *as*

$$f_1(\mathrm{d}) = \gamma(s_1)\gamma(s_2)\gamma(s_3)\gamma(s_4), \tag{4.16}$$

where

$$s_1 = \mathrm{b}, s_2 = \mathrm{a}, s_3 = \mathrm{c}, s_4 = \mathrm{d}, \tag{4.17}$$

$$\gamma(s_1) = \lambda, \gamma(s_2) = 00, \gamma(s_3) = 111, \gamma(s_4) = 11. \tag{4.18}$$

- *Next, we consider the* γ*-decomposition of* $f_0(\mathrm{c})$*. We have* $\mathcal{S}^{\prec}_{F,0}(f_0(\mathrm{c})) = \{\mathrm{a}\}$*. Thus we obtain the* γ*-decomposition as*

$$f_0(\mathrm{c}) = \gamma(s_1)\gamma(s_2), \tag{4.19}$$

where

$$s_1 = \mathrm{a}, s_2 = \mathrm{c}, \tag{4.20}$$

$$\gamma(s_1) = 01, \gamma(s_2) = 00. \tag{4.21}$$

We show the basic properties of γ-decomposition as the following Lemma 4.2.6.

Lemma 4.2.6. *For any* $F(f, \tau) \in \mathscr{F}_1$*,* $i \in [F]$ *and* $s \in \mathcal{S}$*, the following statements (i)–(iii) hold, where* $\gamma(s_1)\gamma(s_2)\ldots\gamma(s_\rho)$ *is the* γ*-decomposition of* $f_i(s)$*.*

(i) $\mathcal{S}_{F,i}(\lambda) \neq \emptyset \iff f_i(s_1) = \gamma(s_1) = \lambda$.

(ii) For any $r = 1, 2, \ldots, \rho$*, if* $r \geq 2$ *or* $|\mathcal{P}^2_{F,i}| = 2$*, then* $|\gamma(s_r)| \geq 2$.

(iii) For any $r = 2, \ldots, \rho$*, we have* $g_1g_2 \in \bar{\mathcal{P}}^2_{F,i}(f_i(s_{r-1}))$*, where* $\gamma(s_r) = g_1g_2\cdots g_l$*.*

Proof of Lemma 4.2.6. (Proof of (i)): Directly from the definition of γ-decomposition.

(Proof of (ii)): We prove for the following two cases separately: the case $r \geq 2$ and the case $r = 1, |\mathcal{P}^2_{F,i}| = 2$.

- The case $r \geq 2$: We have $|\gamma(s_r)| \geq 1$ by (4.13). If we assume $\gamma(s_r) = c$ for some $c \in \mathcal{C}$, then $f_i(s_r) = f_i(s_{r-1})\gamma(s_r) = f_i(s_{r-1})c$ holds, which conflicts with Lemma 4.2.4 (v). This shows $|\gamma(s_r)| \geq 2$ as desired.

- The case $r = 1, |\mathcal{P}^2_{F,i}| = 2$: By Lemma 4.2.4 (ii) (a), we have $|\gamma(s_1)| = |f_i(s_1)| \geq 2$.

(Proof of (iii)): By (ii) of this lemma, we have $|\gamma(s_r)| \geq 2$. Hence, we have $f_i(s_r) = f_i(s_{r-1})\gamma(s_r) \succeq f_i(s_{r-1})g_1g_2$, which leads to $g_1g_2 \in \bar{\mathcal{P}}^2_{F,i}(f_i(s_{r-1}))$ as desired. □

Using γ-decomposition, we now state the definition of $\dot{F}$ as the following Definition 4.2.4.

Definition 4.2.4. *For $F(f,\tau) \in \mathscr{F}_1$, we define $\dot{F}(\dot{f},\dot{\tau}) \in \mathscr{F}^{(|F|)}$ as*

$$\dot{f}_i(s) \coloneqq \dot{\gamma}(s_1)\dot{\gamma}(s_2)\dots\dot{\gamma}(s_\rho), \tag{4.22}$$

$$\dot{\tau}_i(s) \coloneqq \tau_i(s) \tag{4.23}$$

for $i \in [F]$ and $s \in \mathcal{S}$. Here, $\dot{\gamma}(s_r)$ is defined as

$$\dot{\gamma}(s_r) \coloneqq \begin{cases} a_{F,i}g_1g_3g_4\dots g_l & \text{if } r = 1, |\mathcal{P}^2_{F,i}| = 2, \\ \gamma(s_r) & \text{if } r = 1, |\mathcal{P}^2_{F,i}| \geq 3, \\ \bar{a}_{F,\tau_i(s_{r-1})}g_1g_3g_4\dots g_l & \text{if } r \geq 2, |\bar{\mathcal{P}}^1_{F,i}(f_i(s_{r-1}))| = 2, \\ \bar{a}_{F,\tau_i(s_{r-1})}0g_3g_4\dots g_l & \text{if } r \geq 2, |\bar{\mathcal{P}}^1_{F,i}(f_i(s_{r-1}))| = 1, |\bar{\mathcal{P}}^1_{F,\tau_i(s_{r-1})}| = 1, \\ \bar{a}_{F,\tau_i(s_{r-1})}1g_3g_4\dots g_l & \text{if } r \geq 2, |\bar{\mathcal{P}}^1_{F,i}(f_i(s_{r-1}))| = 1, \\ & \qquad |\bar{\mathcal{P}}^1_{F,\tau_i(s_{r-1})}| = 2, |\mathcal{P}^2_{F,\tau_i(s_{r-1})}| = 2, \\ \gamma(s_r) & \text{if } r \geq 2, |\bar{\mathcal{P}}^1_{F,i}(f_i(s_{r-1}))| = 1, \\ & \qquad |\bar{\mathcal{P}}^1_{F,\tau_i(s_{r-1})}| = 2, |\mathcal{P}^2_{F,\tau_i(s_{r-1})}| \geq 3 \end{cases} \tag{4.24}$$

for $r = 1, 2, \dots, \rho$, where $\gamma(s_1)\gamma(s_2)\dots\gamma(s_\rho)$ is the γ-decomposition of $f_i(s)$ and $\gamma(s_r) = g_1g_2\dots g_l$. Also, $a_{F,i} \in \mathcal{C}$ is defined by the following recursive formula:

$$a_{F,i} \coloneqq \begin{cases} a_{F,\tau_i(s')} & \text{if } \mathcal{S}_{F,i}(\lambda) = \{s'\} \text{ for some } s' \in \mathcal{S}', \\ 0 & \text{if } |\mathcal{S}_{F,i}(\lambda)| \neq 1, \mathcal{P}^2_{F,i} \ni 00, \\ 1 & \text{if } |\mathcal{S}_{F,i}(\lambda)| \neq 1, \mathcal{P}^2_{F,i} \not\ni 00, \end{cases} \tag{4.25}$$

and $\bar{a}_{F,i}$ denotes the negation of $a_{F,i}$, that is, $\bar{a}_{F,i} \coloneqq 1 - a_{F,i}$.

We refer to the operation of obtaining the code-tuple $\dot{F}$ from a given code-tuple $F \in \mathscr{F}_1$ as dot operation*.*

Remark 4.2.1. *In Definition 4.2.4, it holds that* $|\gamma(s_r)| < 2$ *only if* $r = 1$ *and* $|\mathcal{P}^2_{F,i}| \geq 3$ *by Lemma 4.2.6 (ii). Hence, the right hand side of (4.24) has enough length so that* $\dot{\gamma}(s_r)$ *is well-defined for every case.*

Example 4.2.4. *We consider* $F(f, \tau) := F^{(\epsilon)}$ *in Table 4.1. Then* $a_{F,i}, i \in [F]$ *are given as follows.*

- $a_{F,0} = 1$ *applying the third case of (4.25) since* $|\mathcal{S}_{F,0}(\lambda)| \neq 1$ *and* $\mathcal{P}^2_{F,0} \not\ni 00$.
- $a_{F,2} = 0$ *applying the second case of (4.25) since* $|\mathcal{S}_{F,2}(\lambda)| \neq 1$ *and* $\mathcal{P}^2_{F,0} \ni 00$.
- $a_{F,1} = a_{F,0} = 1$ *applying the first case of (4.25) since* $|\mathcal{S}_{F,1}(\lambda)| = \{\mathrm{b}\}$.

The codeword $\dot{f}_0(\mathrm{c})$ *is obtained as follows since the* γ*-decomposition of* $f_0(\mathrm{c})$ *is given as (4.19)–(4.21).*

- *we have* $\dot{\gamma}(s_1) = a_{F,0}0 = 10$ *applying the first case of (4.24) since* $|\mathcal{P}^2_{F,0}| = 2$,
- *we have* $\dot{\gamma}(s_2) = \bar{a}_{F,\tau_0(s_1)}0 = \bar{a}_{F,1}0 = 00$ *applying the third case of (4.24) since* $|\bar{\mathcal{P}}^1_{F,0}(f_0(s_1))| = |\bar{\mathcal{P}}^1_{F,0}(01)| = 2$.

Therefore, we obtain $\dot{f}_0(\mathrm{c}) = \dot{\gamma}(s_1)\dot{\gamma}(s_2) = 1000$.

The codeword $\dot{f}_1(\mathrm{d})$ *is obtained as follows since the* γ*-decomposition of* $f_1(\mathrm{d})$ *is given as (4.16)–(4.18).*

- *we have* $\dot{\gamma}(s_1) = \gamma(s_1) = \lambda$ *applying the second case of (4.24) since* $|\mathcal{P}^2_{F,1}| \geq 3$,
- *we have* $\dot{\gamma}(s_2) = \bar{a}_{F,\tau_0(s_1)}1 = \bar{a}_{F,0}1 = 01$ *applying the fifth case of (4.24) since* $|\bar{\mathcal{P}}^1_{F,1}(f_1(s_1))| = |\bar{\mathcal{P}}^1_{F,1}| = 1$, $|\bar{\mathcal{P}}^1_{F,\tau_1(s_1)}| = |\bar{\mathcal{P}}^1_{F,0}| = 2$, *and* $|\mathcal{P}^2_{F,\tau_1(s_1)}| = |\mathcal{P}^2_{F,0}| = 2$,
- *we have* $\dot{\gamma}(s_3) = \bar{a}_{F,\tau_1(s_2)}00 = \bar{a}_{F,1}1 = 001$ *applying the fourth case of (4.24) since* $|\bar{\mathcal{P}}^1_{F,1}(f_1(s_2))| = |\bar{\mathcal{P}}^1_{F,1}(00)| = 1$ *and* $|\bar{\mathcal{P}}^1_{F,\tau_1(s_2)}| = |\bar{\mathcal{P}}^1_{F,1}| = 1$.

Therefore, we obtain $\dot{f}_1(\mathrm{d}) = \dot{\gamma}(s_1)\dot{\gamma}(s_2)\dot{\gamma}(s_3) = 01001$.

The code table $F^{(\zeta)}$ *in Table 4.1 is obtained as* $\dot{F}^{(\epsilon)}$. *Moreover, the code table* $F^{(\eta)}$ *in Table 4.1 is obtained as* $\widehat{F^{(\zeta)}}(= \widehat{\dot{F}^{(\epsilon)}})$.

Now we enumerate some properties of $\dot{F}$ as the following Lemmas 4.2.7–4.2.9.

Lemma 4.2.7. *For any $F(f,\tau) \in \mathscr{F}_1$ and $i \in [F]$, the following statements (i)–(iii) hold.*

(i) Let $s \in \mathcal{S}$ and let $\gamma(s_1)\gamma(s_2)\dots\gamma(s_\rho)$ be the γ-decomposition of $f_i(s)$. Then we have $|\dot{\gamma}(s_r)| = |\gamma(s_r)|$ for any $r = 1, 2, \dots, \rho$.

(ii) For any $s \in \mathcal{S}$, we have $|\dot{f}_i(s)| = |f_i(s)|$.

(iii) For any $s, s' \in \mathcal{S}$, the following equivalence holds: $f_i(s) \preceq f_i(s') \iff \dot{f}_i(s) \preceq \dot{f}_i(s')$.

Proof of Lemma 4.2.7. (Proof of (i)): Directly from (4.24).

(Proof of (ii)): We have

$$\begin{aligned} |\dot{f}_i(s)| &= |\dot{\gamma}(s_1)| + |\dot{\gamma}(s_2)| + \cdots + |\dot{\gamma}(s_\rho)| && (4.26)\\ &\overset{\text{(A)}}{=} |\gamma(s_1)| + |\gamma(s_2)| + \cdots + |\gamma(s_\rho)| && (4.27)\\ &= |f_i(s)|, && (4.28) \end{aligned}$$

where (A) follows from (i) of this lemma.

(Proof of (iii)): See Subsection 4.3.4. □

Lemma 4.2.8. *For any $F(f,\tau) \in \mathscr{F}_1$ and $i \in [F]$, the following statements (i) and (ii) hold.*

(i) (a) If $|\mathcal{P}^2_{F,i}| = 2$, then $\mathcal{P}^2_{\dot{F},i} = \{a_{F,i}0, a_{F,i}1\}$.

(b) For any $s \in \mathcal{S}$, if $|\mathcal{P}^2_{F,j}| \geq 3$, then

$$\mathcal{P}^2_{\dot{F},j} \subseteq \begin{cases} \{00, 01, 10, 11\} & \text{if } |\bar{\mathcal{P}}^1_{F,i}(f_i(s))| = 0,\\ \{a_{F,j}0, a_{F,j}1, \bar{a}_{F,j}1\} & \text{if } |\bar{\mathcal{P}}^1_{F,i}(f_i(s))| = 1, |\bar{\mathcal{P}}^1_{F,j}| = 1,\\ \mathcal{P}^2_{F,j} & \text{if } |\bar{\mathcal{P}}^1_{F,i}(f_i(s))| = 1, |\bar{\mathcal{P}}^1_{F,j}| = 2, \end{cases} \quad (4.29)$$

where $j \coloneqq \tau_i(s) = \dot{\tau}_i(s)$.

(ii) For any $s \in \mathcal{S}$, we have

$$\bar{\mathcal{P}}^2_{\dot{F},i}(\dot{f}_i(s)) \subseteq \begin{cases} \emptyset & \text{if } |\bar{\mathcal{P}}^1_{F,i}(f_i(s))| = 0, \\ \{\bar{a}_{F,j}0, \bar{a}_{F,j}1\} & \text{if } |\bar{\mathcal{P}}^1_{F,i}(f_i(s))| \geq 1, |\mathcal{P}^2_{F,j}| = 2, \\ \{\bar{a}_{F,j}0\} & \text{if } |\bar{\mathcal{P}}^1_{F,i}(f_i(s))| \geq 1, |\mathcal{P}^2_{F,j}| \geq 3, |\bar{\mathcal{P}}^1_{F,j}| = 1, \\ \bar{\mathcal{P}}^2_{F,i}(f_i(s)) & \text{if } |\bar{\mathcal{P}}^1_{F,i}(f_i(s))| \geq 1, |\mathcal{P}^2_{F,j}| \geq 3, |\bar{\mathcal{P}}^1_{F,j}| = 2, \end{cases} \tag{4.30}$$

where $j \coloneqq \tau_i(s) = \dot{\tau}_i(s)$.

See Subsection 4.3.5 for the proof of Lemma 4.2.8.

The next lemma relates to $d_{F,i}$ and $a_{F,i}$ defined in Definitions 3.4.1 and 4.2.4, respectively.

Lemma 4.2.9. *For any $F(f, \tau) \in \mathscr{F}_1$ and $i \in [F]$, the following statements (i) and (ii) hold.*

(i) If $|\mathcal{P}^2_{F,i}| = 2$, then $d_{\dot{F},i} = a_{F,i}$.

(ii) For any $s, s' \in \mathcal{S}$, if $s \neq s'$ and $\dot{f}_i(s) = \dot{f}_i(s')$, then $d_{\dot{F},\dot{\tau}_i(s)} = a_{F,\tau_i(s)} \neq a_{F,\tau_i(s')} = d_{\dot{F},\dot{\tau}_i(s')}$.

See Subsection 4.3.6 for the proof of Lemma 4.2.9.

Using the properties above, we now prove the following Lemma 4.2.10.

Lemma 4.2.10. *For any $F \in \mathscr{F}_1$, we have $\widehat{\dot{F}} \in \mathscr{F}_1$ and $L(\widehat{\dot{F}}) = L(F)$.*

Proof of Lemma 4.2.10. It suffices to prove the following three statements (i)–(iii) for any $F \in \mathscr{F}_1$.

(i) $\widehat{\dot{F}} \in \mathscr{F}_{\text{2-dec}}$.

(ii) $\mathcal{P}^1_{\widehat{\dot{F}},i} = \{0, 1\}$ for any $i \in [F]$.

(iii) $\widehat{\dot{F}} \in \mathscr{F}_{\text{reg}}$ and $L(\widehat{\dot{F}}) = L(F)$.

(Proof of (i)): It suffices to prove $\dot{F} \in \mathscr{F}_{\text{2-dec}}$ because this implies $\widehat{\dot{F}} \in \mathscr{F}_{\text{2-dec}}$ by Lemma 3.4.1 (iv).

We first show that $\dot{F}$ satisfies Definition 2.2.3 (a). Choose $i \in [F]$ and $s \in \mathcal{S}$ arbitrarily and put $j \coloneqq \tau_i(s)$. We consider the following two cases separately: the case $|\bar{\mathcal{P}}^1_{F,i}(f_i(s))| = 0$ and the case $|\bar{\mathcal{P}}^1_{F,i}(f_i(s))| \geq 1$.

- The case $|\bar{\mathcal{P}}^1_{F,i}(f_i(s))| = 0$: We have

$$\mathcal{P}^2_{\dot{F},j} \cap \bar{\mathcal{P}}^2_{\dot{F},i}(\dot{f}_i(s)) \overset{(A)}{\subseteq} \{00, 01, 10, 11\} \cap \bar{\mathcal{P}}^2_{\dot{F},i}(\dot{f}_i(s)) \overset{(B)}{\subseteq} \{00, 01, 10, 11\} \cap \emptyset = \emptyset \tag{4.31}$$

as desired, where (A) follows from $|\bar{\mathcal{P}}^1_{F,i}(f_i(s))| = 0$ and the first case of (4.29), and (B) follows from $|\bar{\mathcal{P}}^1_{F,i}(f_i(s))| = 0$ and the first case of (4.30).

- The case $|\bar{\mathcal{P}}^1_{F,i}(f_i(s))| \geq 1$: We consider the following three cases separately: the case $|\mathcal{P}^2_{F,j}| = 2$, the case $|\mathcal{P}^2_{F,j}| \geq 3, |\bar{\mathcal{P}}^1_{F,j}| = 1$, and the case $|\mathcal{P}^2_{F,j}| \geq 3, |\bar{\mathcal{P}}^1_{F,j}| = 2$.

  - The case $|\mathcal{P}^2_{F,j}| = 2$: We have

$$\mathcal{P}^2_{\dot{F},j} \cap \bar{\mathcal{P}}^2_{\dot{F},i}(\dot{f}_i(s)) \overset{(A)}{=} \{a_{F,j}0, a_{F,j}1\} \cap \bar{\mathcal{P}}^2_{\dot{F},i}(\dot{f}_i(s)) \tag{4.32}$$

$$\overset{(B)}{\subseteq} \{a_{F,j}0, a_{F,j}1\} \cap \{\bar{a}_{F,j}0, \bar{a}_{F,j}1\} \tag{4.33}$$

$$= \emptyset \tag{4.34}$$

    as desired, where (A) follows from $|\mathcal{P}^2_{F,j}| = 2$ and Lemma 4.2.8 (i) (a), and (B) follows from $|\bar{\mathcal{P}}^1_{F,i}(f_i(s))| \geq 1$, $|\mathcal{P}^2_{F,j}| = 2$, and the second case of (4.30).

  - The case $|\mathcal{P}^2_{F,j}| \geq 3$: Then we have $|\bar{\mathcal{P}}^1_{F,i}(f_i(s))| \leq 1$ by Lemma 4.2.2. Combining this with $|\bar{\mathcal{P}}^1_{F,i}(f_i(s))| \geq 1$, we obtain

$$|\bar{\mathcal{P}}^1_{F,i}(f_i(s))| = 1. \tag{4.35}$$

    * The case $|\bar{\mathcal{P}}^1_{F,j}| = 1$: We have

$$\mathcal{P}^2_{\dot{F},j} \cap \bar{\mathcal{P}}^2_{\dot{F},i}(\dot{f}_i(s))$$

$$\overset{(A)}{\subseteq} \{a_{F,j}0, a_{F,j}1, \bar{a}_{F,j}1\} \cap \bar{\mathcal{P}}^2_{\dot{F},i}(\dot{f}_i(s)) \tag{4.36}$$

$$\overset{(B)}{\subseteq} \{a_{F,j}0, a_{F,j}1, \bar{a}_{F,j}1\} \cap \{\bar{a}_{F,j}0\} \tag{4.37}$$

$$= \emptyset, \tag{4.38}$$

      where (A) follows from (4.35), $|\bar{\mathcal{P}}^1_{F,j}| = 1$, and the second case of (4.29), and (B) follows from $|\bar{\mathcal{P}}^1_{F,i}(f_i(s))| \geq 1$, $|\mathcal{P}^2_{F,j}| \geq 3$, $|\bar{\mathcal{P}}^1_{F,j}| = 1$, and the third case of (4.30).

* The case $|\bar{\mathcal{P}}^1_{F,j}| = 2$: We have

$$\mathcal{P}^2_{\dot{F},j} \cap \bar{\mathcal{P}}^2_{\dot{F},i}(\dot{f}_i(s)) \overset{\text{(A)}}{\subseteq} \mathcal{P}^2_{F,j} \cap \bar{\mathcal{P}}^2_{\dot{F},i}(\dot{f}_i(s)) \overset{\text{(B)}}{\subseteq} \mathcal{P}^2_{F,j} \cap \bar{\mathcal{P}}^2_{F,i}(f_i(s)) \overset{\text{(C)}}{=} \emptyset, \tag{4.39}$$

where (A) follows from (4.35), $|\bar{\mathcal{P}}^1_{F,j}| = 2$, and the third case of (4.29), (B) follows from $|\bar{\mathcal{P}}^1_{F,i}(f_i(s))| \geq 1$, $|\mathcal{P}^2_{F,j}| \geq 3$, $|\bar{\mathcal{P}}^1_{F,j}| = 2$, and the fourth case of (4.30), and (C) follows from $F \in \mathscr{F}_{\text{2-dec}}$.

These cases show that $\dot{F}$ satisfies Definition 2.2.3 (a).

Next, we show that $\dot{F}$ satisfies Definition 2.2.3 (b). Choose $i \in [F]$ and $s, s' \in \mathcal{S}$ such that

$$s \neq s', \quad \dot{f}_i(s) = \dot{f}_i(s') \tag{4.40}$$

arbitrarily and put $j := \tau_i(s)$. Since (4.40) and Lemma 4.2.7 (iii) lead to $f_i(s) = f_i(s')$, we have

$$|\mathcal{P}^2_{F,\tau_i(s)}| = |\mathcal{P}^2_{F,\tau_i(s')}| = 2 \tag{4.41}$$

applying Lemma 4.2.4 (iii). Hence, we obtain

$$\mathcal{P}^2_{\dot{F},\tau_i(s)} \cap \mathcal{P}^2_{\dot{F},\tau_i(s')} \overset{\text{(A)}}{=} \{a_{F,\tau_i(s)}0, a_{F,\tau_i(s)}1\} \cap \{a_{F,\tau_i(s')}0, a_{F,\tau_i(s')}1\} \overset{\text{(B)}}{=} \emptyset \tag{4.42}$$

as desired, where (A) follows from (4.41) and Lemma 4.2.8 (i) (a), and (B) follows since $a_{F,\tau_i(s)} \neq a_{F,\tau_i(s')}$ by (4.40) and Lemma 4.2.9 (ii).

(Proof of (ii)): We prove for the following two cases separately: (I) the case $\mathcal{S}_{F,i}(\lambda) = \emptyset$; (II) the case $\mathcal{S}_{F,i}(\lambda) \neq \emptyset$.

(I) The case $\mathcal{S}_{F,i}(\lambda) = \emptyset$: It suffices to show

$$^\forall c \in \mathcal{C}; {}^\exists \boldsymbol{x} \in \mathcal{S}^*; \dot{f}^*_i(\boldsymbol{x}) \succeq d_{\dot{F},i}c \tag{4.43}$$

because this implies that for any $c \in \mathcal{C}$, there exists $\boldsymbol{x} \in \mathcal{S}^*$ such that

$$d_{\dot{F},i}c \overset{\text{(A)}}{\preceq} \dot{f}^*_i(\boldsymbol{x}) \preceq \dot{f}^*_i(\boldsymbol{x}) d_{\dot{F},\tau^*_i(\boldsymbol{x})} \overset{\text{(B)}}{=} d_{\dot{F},i}\widehat{\dot{f}}^*_i(\boldsymbol{x}), \tag{4.44}$$

where (A) follows from (4.43), and (B) follows from Lemma 3.4.1 (i). This shows that $\widehat{\dot{f}}^*_i(\boldsymbol{x}) \succeq c$ for some $\boldsymbol{x} \in \mathcal{S}^*$, which leads to $c \in \mathcal{P}^1_{\hat{\dot{F}},i}$ as desired. Thus, we prove (4.43) considering the following two cases separately: the case $|\mathcal{P}^2_{F,i}| = 2$ and the case $|\mathcal{P}^2_{F,i}| \geq 3$.

- The case $|\mathcal{P}^2_{F,i}| = 2$: For any $c \in \mathcal{C}$, we have

$$\mathcal{P}^2_{\dot{F},i} \overset{\text{(A)}}{=} \{a_{F,i}0, a_{F,i}1\} \overset{\text{(B)}}{=} \{d_{\dot{F},i}0, d_{\dot{F},i}1\} \ni d_{\dot{F},i}c, \tag{4.45}$$

where (A) follows from Lemma 4.2.8 (i) (a), and (B) follows from Lemma 4.2.9 (i). Hence, there exists $\boldsymbol{x} \in \mathcal{S}^+$ such that $\dot{f}^*_{\dot{F},i}(\boldsymbol{x}) \succeq d_{\dot{F},i}c$ as desired.

- The case $|\mathcal{P}^2_{F,i}| \geq 3$: Choose $c \in \mathcal{C}$ arbitrarily. We have $\mathcal{P}^1_{F,i} = \{0,1\} \ni c$ by $F \in \mathscr{F}_1$. Hence, there exists $\boldsymbol{x} = x_1x_2\ldots x_n \in \mathcal{S}^+$ such that $f^*_i(\boldsymbol{x}) \succeq c$. Let $\gamma(s_1)\gamma(s_2)\ldots\gamma(s_\rho)$ be the γ-decomposition of $f_i(x_1)$. We have

$$\dot{f}^*_i(\boldsymbol{x}) \succeq \dot{f}_i(x_1) \succeq \dot{\gamma}(s_1) \overset{\text{(A)}}{=} \gamma(s_1) \overset{\text{(B)}}{\succeq} c, \tag{4.46}$$

where (A) follows from $|\mathcal{P}^2_{F,i}| \geq 3$ and the second case of (4.24), and (B) follows from $\mathcal{S}_{F,i}(\lambda) = \emptyset$ and Lemma 4.2.6 (i).
Since c is arbitrarily chosen, we have $\mathcal{P}^1_{\dot{F},i} = \{0,1\}$ by (4.46). This implies $d_{\dot{F},i} = \lambda$ by (3.123). Therefore, by (4.46), we obtain $\dot{f}^*_i(\boldsymbol{x}) \succeq c = d_{\dot{F},i}c$ for any $c \in \mathcal{C}$ as desired.

(II) The case $\mathcal{S}_{F,i}(\lambda) \neq \emptyset$: By Lemma 2.3.3, we can choose the longest sequence $\boldsymbol{x} \in \mathcal{S}^+$ such that $f^*_i(\boldsymbol{x}) = \lambda$. Then $\mathcal{S}_{F,\tau^*_i(\boldsymbol{x})}(\lambda) = \emptyset$. Hence, from the result of the case (I) above, we have $\mathcal{P}^2_{\hat{\dot{F}},\tau^*_i(\boldsymbol{x})} = \{0,1\}$. Thus, we obtain

$$\mathcal{P}^2_{\hat{\dot{F}},i} \overset{\text{(A)}}{\supseteq} \mathcal{P}^2_{\hat{\dot{F}},\tau^*_i(x_1)} \overset{\text{(A)}}{\supseteq} \mathcal{P}^2_{\hat{\dot{F}},\tau^*_i(x_1x_2)} \overset{\text{(A)}}{\supseteq} \cdots \overset{\text{(A)}}{\supseteq} \mathcal{P}^2_{\hat{\dot{F}},\tau^*_i(\boldsymbol{x})} = \{0,1\} \tag{4.47}$$

as desired, where (A)s follow from Lemma 2.2.1 (i).

(Proof of (iii)): We have

$$Q(F) \overset{\text{(A)}}{=} Q(\dot{F}) \overset{\text{(B)}}{=} Q(\hat{\dot{F}}), \tag{4.48}$$

where (A) follows from (4.23), and (B) follows from (3.122) (cf. Remark 2.4.1). Hence, $F \in \mathscr{F}_{\mathrm{reg}}$ implies $\hat{\dot{F}} \in \mathscr{F}_{\mathrm{reg}}$. Also, we have

$$L(F) \overset{\text{(A)}}{=} L(\dot{F}) \overset{\text{(B)}}{=} L(\hat{\dot{F}}), \tag{4.49}$$

where (A) follows from (4.48) and Lemma 4.2.7 (ii) (cf. Remark 2.4.1), and (B) follows from Lemma 3.4.1 (iii). □

For $F \in \mathscr{F}_1$ and an integer $t \geq 0$, we define

$$F^{(t)} = \begin{cases} F & \text{if } t = 0, \\ \widehat{\dot{F^{(t-1)}}} & \text{if } t > 0. \end{cases} \tag{4.50}$$

Namely, $F^{(t)}$ is the code-tuple obtained by applying dot operation and rotation to F t times. We now prove that any code-tuple of $\mathscr{F}_1$ is transformed into a code-tuple of $\mathscr{F}_2$ by repeating of dot operation and rotation, that is, $\mathcal{M}_{F^{(t)}} = \emptyset$ holds for a sufficiently large t, where $\mathcal{M}_F \coloneqq \{i \in [F] : |\mathcal{P}^2_{F,i}| = 2\}$. To prove this fact, we use the following Lemma 4.2.11. See Subsection 4.3.7 for the proof of Lemma 4.2.11.

Lemma 4.2.11. *For any $F \in \mathscr{F}_{\text{2-opt}} \cap \mathscr{F}_1 \cap \mathscr{F}_{\text{irr}}$ and two integers t and t' such that $0 \leq t < t'$, it holds that $\mathcal{M}_{F^{(t)}} \cap \mathcal{M}_{F^{(t')}} = \emptyset$.*

Lemma 4.2.12. *$\mathscr{F}_{\text{2-opt}} \cap \mathscr{F}_2 \neq \emptyset$.*

Proof of Lemma 4.2.12. By Lemma 4.2.3, there exists $F \in \mathscr{F}_{\text{2-opt}} \cap \mathscr{F}_1$. By Lemmas 2.5.3 and 2.5.4, we may assume $F \in \mathscr{F}_{\text{irr}}$ without loss of generality. Consider $|F| + 1$ code-tuples $F^{(0)}, F^{(1)}, \ldots, F^{(|F|)}$. Because Lemma 4.2.11 shows that the $|F| + 1$ sets $\mathcal{M}_{F^{(0)}}, \mathcal{M}_{F^{(1)}}, \ldots, \mathcal{M}_{F^{(|F|)}}$ are disjoint, there exists an integer $\bar{t} \in \{0, 1, 2, \ldots, |F|\}$ such that $\mathcal{M}_{F^{(\bar{t})}} = \emptyset$. This shows that $|\mathcal{P}^2_{F^{(\bar{t})},i}| \geq 3$ for any $i \in [F]$. Since $F^{(\bar{t})} \in \mathscr{F}_{\text{2-opt}} \cap \mathscr{F}_1$ by Lemma 4.2.10, we obtain $F^{(\bar{t})} \in \mathscr{F}_{\text{2-opt}} \cap \mathscr{F}_2$. □

We state some properties of $\mathscr{F}_2$ as the following Lemmas 4.2.13 and 4.2.14.

Lemma 4.2.13. *For any $F(f, \tau) \in \mathscr{F}_2$ and $i \in [F]$, the mapping f_i is injective.*

Proof of Lemma 4.2.13. For any $s \in \mathcal{S}$, we have

$$|\mathcal{S}_{F,i}(f_i(s))| = \frac{3|\mathcal{S}_{F,i}(f_i(s))|}{3} \overset{\text{(A)}}{\leq} \frac{\sum_{s' \in \mathcal{S}_{F,i}(f_i(s))} |\mathcal{P}^2_{F,\tau_i(s')}|}{3} \overset{\text{(B)}}{\leq} \frac{|\mathcal{P}^2_{F,i}(f_i(s))|}{3} \leq \frac{4}{3}, \tag{4.51}$$

where (A) follows since $|\mathcal{P}^2_{F,\tau_i(s')}| \geq 3$ for any $s' \in \mathcal{S}_{F,i}(f_i(s))$ from $F \in \mathscr{F}_2$, and (B) follows from Lemma 2.2.3. Therefore, we have $|\mathcal{S}_{F,i}(f_i(s))| \leq 1$ for any $s \in \mathcal{S}$. This shows that f_i is injective as desired. □

Lemma 4.2.14. *For any $F(f, \tau) \in \mathscr{F}_{\text{2-opt}} \cap \mathscr{F}_2$, there exists $i \in \mathcal{R}_F$ such that $|\mathcal{P}^2_{F,i}| = 4$.*

Proof of Lemma 4.2.14. Choose $p \in \mathcal{R}_F$. By Lemma 2.2.2 (ii), there exists $s \in \mathcal{S}$ such that $\bar{\mathcal{P}}^0_{F,p}(f_p(s)) = \emptyset$. Also, by Lemma 4.2.13, we have $|\mathcal{S}_{F,p}(f_p(s))| = 1$. Hence, by Lemma 4.2.5, we obtain $|\mathcal{P}^2_{F,i}| = 4$ for $i \coloneqq \tau_p(s)$.

By $p \in \mathcal{R}_F$, for any $j \in [F]$, there exists $\boldsymbol{x} \in \mathcal{S}^*$ such that $\tau^*_j(\boldsymbol{x}) = p$, which leads to

$$\tau^*_j(\boldsymbol{x}s) \overset{\text{(A)}}{=} \tau_{\tau^*_j(\boldsymbol{x})}(s) = \tau_p(s) = i, \tag{4.52}$$

where (A) follows from Lemma 2.1.1 (ii). This shows $i \in \mathcal{R}_F$. □

4.2.3 The Class $\mathscr{F}_3$

In this subsection, we prove $\mathscr{F}_{\text{2-opt}} \cap \mathscr{F}_3 \neq \emptyset$, which proof is outlined as follows.

- First, we define the code-tuple $\ddot{F}$ as Definition 4.2.5 for a given code-tuple $F \in \mathscr{F}_2$.
- Then we show that $\ddot{F} \in \mathscr{F}_{\text{2-opt}} \cap \mathscr{F}_3$ holds for any $F \in \mathscr{F}_{\text{2-opt}} \cap \mathscr{F}_2$. This shows $\mathscr{F}_{\text{2-opt}} \cap \mathscr{F}_3 \neq \emptyset$ since $\mathscr{F}_{\text{2-opt}} \cap \mathscr{F}_2 \neq \emptyset$ by Lemma 4.2.12.

Definition 4.2.5. *For $F(f, \tau) \in \mathscr{F}_2$, we define $\ddot{F}(\ddot{f}, \ddot{\tau}) \in \mathscr{F}^{(|F|)}$ as*

$$\ddot{f}_i(s) \coloneqq \ddot{\gamma}(s_1)\ddot{\gamma}(s_2)\dots\ddot{\gamma}(s_\rho), \tag{4.53}$$

$$\ddot{\tau}_i(s) \coloneqq \tau_i(s) \tag{4.54}$$

for $i \in [F]$ and $s \in \mathcal{S}$. Here, $\ddot{\gamma}(s_r)$ is defined as

$$\ddot{\gamma}(s_r) = \begin{cases} \gamma(s_r) & \text{if } r = 1, |\mathcal{P}^2_{F,i}| = 4, \\ 1 & \text{if } r = 1, |\mathcal{P}^2_{F,i}| = 3, |\gamma(s_r)| = 1, \\ 01g_3g_4\dots g_l & \text{if } r = 1, |\mathcal{P}^2_{F,i}| = 3, |\gamma(s_r)| \geq 2, g_1\bar{g}_2 \notin \mathcal{P}^2_{F,i}, \\ 1g_2g_3g_4\dots g_l & \text{if } r = 1, |\mathcal{P}^2_{F,i}| = 3, |\gamma(s_r)| \geq 2, g_1\bar{g}_2 \in \mathcal{P}^2_{F,i}, \\ 00g_3g_4\dots g_l & \text{if } r \geq 2 \end{cases} \tag{4.55}$$

for $r = 1, 2, \dots, \rho$, where $\gamma(s_1)\gamma(s_2)\dots\gamma(s_\rho)$ is the γ-decomposition of $f_i(s)$ and $\gamma(s_r) = g_1g_2\dots g_l$.

Example 4.2.5. *We consider $F(f, \tau) \coloneqq F^{(\eta)}$ in Table 4.1.*

- *The* γ*-decomposition of* $f_0(\mathrm{d})$ *is* $f_0(\mathrm{d}) = \gamma(s_1)$*, where* $\gamma(s_1) = 001$*. We have* $\ddot{\gamma}(s_1) = \gamma(s_1) = 001$ *applying the first case of (4.55) since* $|\mathcal{P}^2_{F,0}| = 4$*. Hence, we have* $\ddot{f}_0(\mathrm{d}) = \ddot{\gamma}(s_1) = 001$*.*

- *The* γ*-decomposition of* $f_1(\mathrm{c})$ *is* $f_1(\mathrm{c}) = \gamma(s_1)\gamma(s_2)$*, where* $\gamma(s_1) = 01$ *and* $\gamma(s_2) = 001$*. We have* $\ddot{\gamma}(s_1) = 01$ *applying the third case of (4.55) since* $|\mathcal{P}^2_{F,1}| = 3$ *and* $00 \notin \mathcal{P}^2_{F,1}$*. We have* $\ddot{\gamma}(s_2) = 001$ *applying the fifth case of (4.55). Hence, we have* $\ddot{f}_1(\mathrm{c}) = \ddot{\gamma}(s_1)\ddot{\gamma}(s_2) = 01001$*.*

- *The* γ*-decomposition of* $f_1(\mathrm{b})$ *is* $f_1(\mathrm{b}) = \gamma(s_1)$*, where* $\gamma(s_1) = 1$*. We have* $\ddot{\gamma}(s_1) = 1$ *applying the second case of (4.55) since* $|\mathcal{P}^2_{F,1}| = 3$ *and* $|\gamma(s_1)| = 1$*. Hence, we have* $\ddot{f}_1(\mathrm{b}) = \ddot{\gamma}(s_1) = 1$*.*

- *The* γ*-decomposition of* $f_2(\mathrm{d})$ *is* $f_2(\mathrm{d}) = \gamma(s_1)$*, where* $\gamma(s_1) = 011$*. We have* $\ddot{\gamma}(s_1) = 111$ *applying the fourth case of (4.55) since* $|\mathcal{P}^2_{F,2}| = 3$ *and* $01 \in \mathcal{P}^2_{F,2}$*. Hence, we have* $\ddot{f}_2(\mathrm{d}) = \ddot{\gamma}(s_1) = 111$*.*

The code table $F^{(\theta)}$ *in Table 4.1 is obtained as* $\ddot{F}^{(\eta)}$*.*

We state some properties of $\ddot{F}$ as the following Lemmas 4.2.15 and 4.2.16 (cf. Lemmas 4.2.7 and 4.2.8).

Lemma 4.2.15. *For any* $F(f,\tau) \in \mathscr{F}_2$ *and* $i \in [F]$*, the following statements (i)–(iii) hold.*

(i) Let $s \in \mathcal{S}$ *and let* $\gamma(s_1)\gamma(s_2)\ldots\gamma(s_\rho)$ *be the* γ*-decomposition of* $f_i(s)$*. Then we have* $|\ddot{\gamma}(s_r)| = |\gamma(s_r)|$ *for any* $r = 1, 2, \ldots, \rho$*.*

(ii) For any $s \in \mathcal{S}$*, we have* $|\ddot{f}_i(s)| = |f_i(s)|$*.*

(iii) For any $s, s' \in \mathcal{S}$*, the following equivalence holds:* $f_i(s) \preceq f_i(s') \iff \ddot{f}_i(s) \preceq \ddot{f}_i(s')$*.*

Proof of Lemma 4.2.15. (Proof of (i)): Directly from (4.55).

(Proof of (ii)): We have

$$\begin{aligned} |\ddot{f}_i(s)| &= |\ddot{\gamma}(s_1)| + |\ddot{\gamma}(s_2)| + \cdots + |\ddot{\gamma}(s_\rho)| && (4.56)\\ &\overset{\text{(A)}}{=} |\gamma(s_1)| + |\gamma(s_2)| + \cdots + |\gamma(s_\rho)| && (4.57)\\ &= |f_i(s)|, && (4.58) \end{aligned}$$

where (A) follows from (i) of this lemma.

(Proof of (iii)): See Subsection 4.3.8. □

Lemma 4.2.16. *For any $F \in \mathscr{F}_2$ and $i \in [F]$, the following statements (i) and (ii) hold.*

(i)

$$\mathcal{P}^2_{\ddot{F},i} = \begin{cases} \{01, 10, 11\} & \textit{if } |\mathcal{P}^2_{F,i}| = 3, \\ \{00, 01, 10, 11\} & \textit{if } |\mathcal{P}^2_{F,i}| = 4. \end{cases} \tag{4.59}$$

(ii) For any $s \in \mathcal{S}$, we have

$$\bar{\mathcal{P}}^2_{\ddot{F},i}(\ddot{f}_i(s)) = \begin{cases} \emptyset & \textit{if } \bar{\mathcal{P}}^0_{F,i}(f_i(s)) = \emptyset, \\ \{00\} & \textit{if } \bar{\mathcal{P}}^0_{F,i}(f_i(s)) \neq \emptyset. \end{cases} \tag{4.60}$$

See Subsection 4.3.9 for the proof of Lemma 4.2.16.

Using the properties above, we prove the main result of this subsection as the following Lemma 4.2.17.

Lemma 4.2.17. *$\mathscr{F}_{\text{2-opt}} \cap \mathscr{F}_3 \neq \emptyset$.*

Proof of Lemma 4.2.17. By Lemma 4.2.12, there exists $F(f, \tau) \in \mathscr{F}_2 \cap \mathscr{F}_{\text{2-opt}}$. We have

$$Q(\ddot{F}) = Q(F) \tag{4.61}$$

by (4.54) (cf. Remark 2.4.1).

Now, we show $\ddot{F} \in \mathscr{F}_{\text{2-opt}} \cap \mathscr{F}_3$ as follows.

- (Proof of $\ddot{F} \in \mathscr{F}_{\text{reg}}$): From $F \in \mathscr{F}_2 \subseteq \mathscr{F}_{\text{reg}}$ and (4.61).
- (Proof of $\ddot{F} \in \mathscr{F}_{\text{2-dec}}$): We first show that $\ddot{F}$ satisfies Definition 2.2.3 (a). We choose $i \in [\ddot{F}]$ and $s \in \mathcal{S}$ arbitrarily and consider the following two cases separately: the case $\bar{\mathcal{P}}^0_{F,i}(f_i(s)) = \emptyset$ and the case $\bar{\mathcal{P}}^0_{F,i}(f_i(s)) \neq \emptyset$.
  - The $\bar{\mathcal{P}}^0_{F,i}(f_i(s)) = \emptyset$: We have

$$\mathcal{P}^2_{\ddot{F},\ddot{\tau}_i(s)} \cap \bar{\mathcal{P}}^2_{\ddot{F},i}(\ddot{f}_i(s)) \overset{\text{(A)}}{=} \mathcal{P}^2_{\ddot{F},\ddot{\tau}_i(s)} \cap \emptyset = \emptyset, \tag{4.62}$$

    where (A) follows from $\bar{\mathcal{P}}^0_{F,i}(f_i(s)) = \emptyset$ and the first case of (4.60).

– The case $\bar{\mathcal{P}}^0_{F,i}(f_i(s)) \neq \emptyset$: By Lemma 4.2.2, we have $|\mathcal{P}^2_{F,\tau_i(s)}| \leq 3$. In particular, it holds that

$$|\mathcal{P}^2_{F,\tau_i(s)}| = 3 \tag{4.63}$$

by $F \in \mathscr{F}_2$. Thus, we have

$$\begin{aligned} \mathcal{P}^2_{\ddot{F},\ddot{\tau}_i(s)} \cap \bar{\mathcal{P}}^2_{\ddot{F},i}(\ddot{f}_i(s)) &\overset{\text{(A)}}{=} \{01,10,11\} \cap \bar{\mathcal{P}}^2_{\ddot{F},i}(\ddot{f}_i(s)) && (4.64)\\ &\overset{\text{(B)}}{=} \{01,10,11\} \cap \{00\} && (4.65)\\ &= \emptyset, && (4.66) \end{aligned}$$

where (A) follows from (4.63) and the first case of (4.59), and (B) follows from $\bar{\mathcal{P}}^0_{F,i}(f_i(s)) \neq \emptyset$ and the second case of (4.60).

These cases show that $\ddot{F}$ satisfies Definition 2.2.3 (a).

Also, by $F \in \mathscr{F}_2$ and Lemma 4.2.13, all the mappings $f_0, f_1, \ldots, f_{|F|-1}$ are injective. This proves that $\ddot{F}$ satisfies Definition 2.2.3 (b) (cf. Remark 2.2.1).

- (Proof of $\ddot{F} \in \mathscr{F}_{\text{2-opt}}$): For any $i \in [F]$, we have $L_i(\ddot{F}) = L_i(F)$ by Lemma 4.2.15 (ii) and we have $\pi_i(\ddot{F}) = \pi_i(F)$ by (4.61) (cf. Remark 2.4.1). Hence, we have $L(\ddot{F}) = L(F)$, which leads to $\ddot{F} \in \mathscr{F}_{\text{2-opt}}$ by $F \in \mathscr{F}_{\text{2-opt}}$.
- (Proof of ${}^\forall i \in [\ddot{F}]; \mathcal{P}^2_{\ddot{F},i} \supseteq \{01,10,11\}$): Choose $i \in [\ddot{F}]$ arbitrarily. Since $|\mathcal{P}^2_{F,i}| \geq 3$ by $F \in \mathscr{F}_2$, we obtain $\mathcal{P}^2_{\ddot{F},i} \supseteq \{01,10,11\}$ applying Lemma 4.2.16 (i).

□

4.2.4 The Class $\mathscr{F}_4$

Lemma 4.2.18. $\mathscr{F}_{\text{2-opt}} \cap \mathscr{F}_4 \neq \emptyset$.

Proof of Lemma 4.2.18. By Lemma 4.2.17, there exists $F \in \mathscr{F}_{\text{2-opt}} \cap \mathscr{F}_3$. Applying Theorem 3.1.1, there exists $F^\dagger(f^\dagger, \tau^\dagger) \in \mathscr{F}_{\text{2-opt}} \cap \mathscr{F}_3$ satisfying $|F^\dagger| = |\mathscr{P}^2_{F^\dagger}|$. By Lemma 4.2.14, there exists $i \in \mathcal{R}_{F^\dagger}$ such that $\mathcal{P}^2_{F^\dagger,i} = \{00,01,10,11\}$. Hence, $F^\dagger$ satisfies exactly one of the following conditions (a) and (b).

(a) $|F^\dagger| = 2, \mathcal{P}^2_{F^\dagger,0} = \{00, 01, 10, 11\}, \mathcal{P}^2_{F^\dagger,1} = \{01, 10, 11\}$ (by swapping the indices of $(f^\dagger_0, \tau^\dagger_0)$ and $(f^\dagger_1, \tau^\dagger_1)$ if necessary).

(b) $|F^\dagger| = 1, \mathcal{P}^2_{F^\dagger,0} = \{00, 01, 10, 11\}$.

In the case (a), we have $F^\dagger \in \mathscr{F}_{\text{2-opt}} \cap \mathscr{F}_4$ as desired. In the case (b), we can see that the code-tuple $F'(f', \tau') \in \mathscr{F}^{(2)}$ defined as below satisfies $F' \in \mathscr{F}_{\text{2-opt}} \cap \mathscr{F}_4$ as desired:

$$f'_0(s_r) \coloneqq f^\dagger_0(s_r), \quad \tau'_0(s_r) \coloneqq \tau^\dagger_0(s_r), \tag{4.67}$$

$$f'_1(s_r) = \begin{cases} 01 & \text{if } r = 1, \\ 1^{r-1}0 & \text{if } 2 \leq r \leq \sigma - 1, \\ 1^{\sigma-1} & \text{if } r = \sigma, \end{cases} \quad \tau'_1(s_r) = 0 \tag{4.68}$$

for $s_r \in \mathcal{S}$, where we suppose $\mathcal{S} = \{s_1, s_2, \ldots, s_\sigma\}$ and the notation 1^l denotes the sequence obtained by concatenating l copies of 1 for an integer $l \geq 1$. □

4.2.5 Proof of Theorem 4.2.1

Finally, we prove the desired Theorem 4.2.1 as follows.

Proof of Theorem 4.2.1. By Lemma 4.2.18, there exists $F \in \mathscr{F}_{\text{2-opt}} \cap \mathscr{F}_4$. We have $0 \in \mathcal{R}_F$ by Lemma 4.2.14. We consider the following two cases separately: the case $\mathcal{R}_F = \{0, 1\}$ and the case $\mathcal{R}_F = \{0\}$.

- The case $\mathcal{R}_F = \{0, 1\}$: We prove $F \in \mathscr{F}_{\text{AIFV}}$ by showing that F satisfies Definition 4.2.1 (i)–(vii).
  - (Proof of (i)): Directly from Lemma 4.2.13.
  - (Proof of (ii)): Choose $s \in \mathcal{S}$ arbitrarily. We first prove $\bar{\mathcal{P}}^1_{F,i}(f_i(s)) \not\ni 1$ by contradiction assuming $\bar{\mathcal{P}}^1_{F,i}(f_i(s)) \ni 1$. Then by Lemma 2.3.2 (ii), we have
  $$\bar{\mathcal{P}}^2_{F,i}(f_i(s)) \ni 1c \tag{4.69}$$
  for some $c \in \mathcal{C}$. On the other hand, by $F \in \mathscr{F}_4$, we have
  $$\mathcal{P}^2_{F,\tau_i(s)} \ni 10, 11. \tag{4.70}$$
  By (4.69) and (4.70), we obtain $\mathcal{P}^2_{F,\tau_i(s)} \cap \bar{\mathcal{P}}^2_{F,i}(f_i(s)) \neq \emptyset$, which leads to $F \notin \mathscr{F}_{\text{2-dec}}$. This conflicts with $F \in \mathscr{F}_4 \subseteq \mathscr{F}_{\text{2-dec}}$.

Next, we prove $\bar{\mathcal{P}}^1_{F,i}(f_i(s)0) \not\ni 1$ by contradiction assuming

$$\bar{\mathcal{P}}^1_{F,i}(f_i(s)0) \ni 1. \tag{4.71}$$

Then we have

$$\begin{aligned}
\mathcal{P}^2_{F,\tau_i(s)} \cap \bar{\mathcal{P}}^2_{F,i}(f_i(s)) &\overset{\text{(A)}}{\supseteq} \mathcal{P}^2_{F,\tau_i(s)} \cap 0\mathcal{P}^1_{F,i}(f_i(s)0) && (4.72)\\
&\overset{\text{(B)}}{\supseteq} \mathcal{P}^2_{F,\tau_i(s)} \cap 0\bar{\mathcal{P}}^1_{F,i}(f_i(s)0) && (4.73)\\
&\overset{\text{(C)}}{\supseteq} \mathcal{P}^2_{F,\tau_i(s)} \cap 0\{1\}, && (4.74)\\
&\overset{\text{(D)}}{\supseteq} \{01, 10, 11\} \cap \{01\} && (4.75)\\
&= \{01\} && (4.76)\\
&\neq \emptyset && (4.77)
\end{aligned}$$

where (A) follows from Lemma 2.2.1 (ii), (B) follows from Lemma 2.2.1 (i), (C) follows from (4.71), and (D) follows from $F \in \mathscr{F}_4 \subseteq \mathscr{F}_3$. Hence, we obtain $F \notin \mathscr{F}_{\text{2-dec}}$, which conflicts with $F \in \mathscr{F}_4 \subseteq \mathscr{F}_{\text{2-dec}}$.

- (Proof of (iii)): Directly from Lemma 4.2.4 (v).
- (Proof of (iv)): Choose $i \in [F]$ and $s \in \mathcal{S}$ arbitrarily and consider the following two cases separately: the case $\bar{\mathcal{P}}^0_{F,i}(f_i(s)) = \emptyset$ and the case $\bar{\mathcal{P}}^0_{F,i}(f_i(s)) \neq \emptyset$:
  * The case $\bar{\mathcal{P}}^0_{F,i}(f_i(s)) = \emptyset$: We have $|\mathcal{P}^2_{F,\tau_i(s)}| = 4$ applying Lemma 4.2.5 since $i \in \{0, 1\} = \mathcal{R}_F$ holds and f_i is injective by Lemma 4.2.13. Hence, we obtain $\tau_i(s) = 0$ by $F \in \mathscr{F}_4$.
  * The case $\bar{\mathcal{P}}^0_{F,i}(f_i(s)) \neq \emptyset$: We have $|\mathcal{P}^2_{F,\tau_i(s)}| \leq 3$ by Lemma 4.2.2, Hence, we obtain $\tau_i(s) = 1$ by $F \in \mathscr{F}_4$.
- (Proof of (v)): We choose $i \in [F]$ arbitrarily and prove that if $f_i(s) = \lambda$ or $f_i(s) = 0$ for some $s \in \mathcal{S}$, then $\mathcal{P}^2_{F,i} \neq \{01, 10, 11\}$, which is equivalent to $i = 0$. Choose $s \in \mathcal{S}$ such that $f_i(s) = \lambda$ or $f_i(s) = 0$. We consider the following two cases separately: the case $f_i(s) = \lambda$ and the case $f_i(s) = 0$.
  * The case $f_i(s) = \lambda$: By Lemma 4.2.13, the mapping f_i is injective. Thus, by Lemma 2.2.2 (iii), we have $\bar{\mathcal{P}}^0_{F,i} \neq \emptyset$.

Hence, by Corollary 2.3.1 (ii) (a), we have

$$\bar{\mathcal{P}}^2_{F,i} \neq \emptyset. \tag{4.78}$$

Also, we have

$$\bar{\mathcal{P}}^2_{F,i} \overset{\text{(A)}}{\subseteq} \mathcal{C}^2 \setminus \mathcal{P}^2_{F,i} \overset{\text{(B)}}{\subseteq} \mathcal{C}^2 \setminus \{01, 10, 11\} = \{00\}, \tag{4.79}$$

where (A) follows from $F \in \mathscr{F}_4 \subseteq \mathscr{F}_{2\text{-dec}}$, and (B) follows from $F \in \mathscr{F}_4 \subseteq \mathscr{F}_3$. Thus, we obtain

$$\mathcal{P}^2_{F,i} \overset{\text{(A)}}{\supseteq} \bar{\mathcal{P}}^2_{F,i} \overset{\text{(B)}}{=} \{00\}. \tag{4.80}$$

where (A) follows from Lemma 2.2.1 (i), and (B) follows from (4.78) and (4.79). This shows $\mathcal{P}^2_{F,i} \neq \{01, 10, 11\}$ as desired.

* The case $f_i(s) = 0$: We have

$$\begin{aligned}
\mathcal{P}^2_{F,i} &\overset{\text{(A)}}{\supseteq} \bar{\mathcal{P}}^2_{F,i} && (4.81)\\
&\overset{\text{(B)}}{\supseteq} 0\mathcal{P}^1_{F,i}(0) && (4.82)\\
&\overset{\text{(C)}}{=} 0\mathcal{P}^1_{F,i}(f_i(s)) && (4.83)\\
&\overset{\text{(D)}}{\supseteq} 0\mathcal{P}^1_{F,\tau_i(s)} && (4.84)\\
&\overset{\text{(E)}}{=} 0\{0, 1\} && (4.85)\\
&\ni 00, && (4.86)
\end{aligned}$$

where (A) follows from Lemma 2.2.1 (i), (B) follows from Lemma 2.2.1 (ii), (C) follows from $f_i(s) = 0$, (D) follows from Lemma 2.2.1 (i), and (E) follows from $F \in \mathscr{F}_4 \subseteq \mathscr{F}_1$. This leads to $\mathcal{P}^2_{F,i} \neq \{01, 10, 11\}$.

– (Proof of (vi)): We prove by contradiction assuming $\bar{\mathcal{P}}^1_{F,1}(0) \ni 0$. We have

$$\mathcal{P}^2_{F,1} \overset{\text{(A)}}{\supseteq} \bar{\mathcal{P}}^2_{F,1} \overset{\text{(B)}}{\supseteq} 0\mathcal{P}^1_{F,1}(0) \overset{\text{(C)}}{\supseteq} 0\bar{\mathcal{P}}^1_{F,1}(0) \overset{\text{(D)}}{\ni} 00, \tag{4.87}$$

where (A) follows from Lemma 2.2.1 (i), (B) follows from Lemma 2.2.1 (ii), (C) follows from Lemma 2.2.1 (i), and (D) follows from $\bar{\mathcal{P}}^1_{F,1}(0) \ni 0$. This shows $\mathcal{P}^2_{F,1} \neq \{01, 10, 11\}$, which conflicts with $F \in \mathscr{F}_4$.

– (Proof of (vii)): We prove by contradiction assuming that there exist $i \in [F]$ and $\boldsymbol{b} \in \mathcal{C}^*$ such that all of the following conditions (a)–(c) hold.

(a) $|\bar{\mathcal{P}}^1_{F,i}(\boldsymbol{b})| = 1$.

(b) $f_i(s)\boldsymbol{c} \neq \boldsymbol{b}$ for any $s \in \mathcal{S}$ and $\boldsymbol{c} \in \mathcal{C}^0 \cup \mathcal{C}^1$.

(c) $(i, \boldsymbol{b}) \neq (1, 0)$.

We have

$$|\mathcal{P}^1_{F,i}(\boldsymbol{b})| \overset{\text{(A)}}{=} |\bar{\mathcal{P}}^1_{F,i}(\boldsymbol{b})| + \sum_{s \in \mathcal{S}_{F,i}(\boldsymbol{b})} |\mathcal{P}^1_{F,\tau_i(s)}| \overset{\text{(B)}}{=} |\bar{\mathcal{P}}^1_{F,i}(\boldsymbol{b})| \overset{\text{(C)}}{=} 1, \quad (4.88)$$

where (A) follows from Lemma 2.2.3, (B) follows since $\mathcal{S}_{F,i}(\boldsymbol{b}) = \emptyset$ by the condition (b), and (C) follows from the condition (a).

We consider the following three cases separately: the case $|\boldsymbol{b}| = 0$, the case $|\boldsymbol{b}| = 1$, and the case $|\boldsymbol{b}| \geq 2$.

* The case $|\boldsymbol{b}| = 0$: By (4.88), we have $|\mathcal{P}^1_{F,i}| = |\mathcal{P}^1_{F,i}(\boldsymbol{b})| = 1$, which conflicts with $F \in \mathscr{F}_4 \subseteq \mathscr{F}_1$.
* The case $|\boldsymbol{b}| = 1$: We have

  $$\mathcal{P}^2_{F,i} \overset{\text{(A)}}{=} \bar{\mathcal{P}}^2_{F,i} \cup \Bigg(\bigcup_{s \in \mathcal{S}_{F,i}(\lambda)} \mathcal{P}^2_{F,\tau_i(s)} \Bigg) \overset{\text{(B)}}{=} \bar{\mathcal{P}}^2_{F,i} \overset{\text{(C)}}{=} 0\mathcal{P}^1_{F,i}(0) \cup 1\mathcal{P}^1_{F,i}(1), \quad (4.89)$$

  where (A) follows from Lemma 2.2.1 (i), (B) follows because $\mathcal{S}_{F,i}(\lambda) = \emptyset$ by $|\boldsymbol{b}| = 1$ and the condition (b), and (C) follows from Lemma 2.2.1 (ii).
  On the other hand, we have $\mathcal{P}^2_{F,0} = \{00, 01, 10, 11\}$ and $\mathcal{P}^2_{F,1} = \{01, 10, 11\}$ by $F \in \mathscr{F}_4$. Hence, comparing with (4.89), we have $\mathcal{P}^1_{F,0}(0) = \mathcal{P}^1_{F,0}(1) = \mathcal{P}^1_{F,1}(1) = \{0, 1\}$ and $\mathcal{P}^1_{F,1}(0) = \{1\}$. Therefore, by (4.88) and $|\boldsymbol{b}| = 1$, it must hold that $(i, \boldsymbol{b}) = (1, 0)$, which conflicts with the condition (c).
* The case $|\boldsymbol{b}| \geq 2$: By the condition (a), we have

  $$\bar{\mathcal{P}}^1_{F,i}(\boldsymbol{b}) = \{a\} \quad (4.90)$$

  for some $a \in \mathcal{C}$. Then there exists $\boldsymbol{x} \in \mathcal{S}^+$ such that

  $$f^*_i(\boldsymbol{x}) \succeq \boldsymbol{b}a, \quad f_i(x_1) \succ \boldsymbol{b}. \quad (4.91)$$

Hence, by $|\boldsymbol{b}| \geq 2$, we have $f_i(x_1) \succ b_1 b_2$, which leads to

$$b_1 b_2 \in \mathcal{P}^2_{F,i}, \tag{4.92}$$

where $b_1 b_2$ denotes the prefix of length 2 of $\boldsymbol{b}$. By $i \in \{0,1\} = \mathcal{R}_F$ and (4.92), we have $\boldsymbol{b}\bar{a} \in \mathcal{P}^*_{F,i}$ applying Lemma 4.3.2 stated in Subsection 4.3.3. Hence, there exists $\boldsymbol{y} \in \mathcal{S}^+$ such that

$$f^*_i(\boldsymbol{y}) \succeq \boldsymbol{b}\bar{a}. \tag{4.93}$$

Then exactly one of $f_i(y_1) \succ \boldsymbol{b}$ and $f_i(y_1) \preceq \boldsymbol{b}$ holds. Now, the latter $f_i(y_1) \preceq \boldsymbol{b}$ holds because the former $f_i(y_1) \succ \boldsymbol{b}$ implies $\bar{a} \in \bar{\mathcal{P}}^1_{F,i}(\boldsymbol{b})$ by (4.93), which conflicts with (4.90). Therefore, there exists $\boldsymbol{c} = c_1 c_2 \ldots c_l \in \mathcal{C}^*$ such that $f_i(y_1)\boldsymbol{c} = \boldsymbol{b}$. By the condition (b), we have $|\boldsymbol{c}| \geq 2$ so that

$$f_i(y_1) c_1 c_2 \preceq \boldsymbol{b}. \tag{4.94}$$

We have

$$f_i(y_1) f^*_{\tau_i(y_1)}(\mathrm{suff}(\boldsymbol{y})) = f^*_i(\boldsymbol{y}) \overset{\text{(A)}}{\succeq} \boldsymbol{b}\bar{a} \succeq \boldsymbol{b} \overset{\text{(B)}}{\succeq} f_i(y_1) c_1 c_2, \tag{4.95}$$

where (A) follows from (4.93), and (B) follows from (4.94). Comparing both sides, we obtain $f^*_{\tau_i(y_1)}(\mathrm{suff}(\boldsymbol{y})) \succeq c_1 c_2$, which leads to

$$c_1 c_2 \in \mathcal{P}^2_{F,\tau_i(y_1)}. \tag{4.96}$$

Also, by (4.91) and (4.94), we have $f_i(x_1) \succ f_i(y_1) c_1 c_2$, which leads to

$$c_1 c_2 \in \bar{\mathcal{P}}^2_{F,i}(f_i(y_1)). \tag{4.97}$$

By (4.96) and (4.97), we obtain $\bar{\mathcal{P}}^2_{F,i}(f_i(y_1)) \cap \mathcal{P}^2_{F,\tau_i(y_1)} \neq \emptyset$, which conflicts with $F \in \mathscr{F}_{2\text{-dec}}$.

- The case $\mathcal{R}_F = \{0\}$: We define $F'(f', \tau') \in \mathscr{F}^{(2)}$ as

$$f'_0(s_r) \coloneqq f_0(s_r), \quad \tau'_0(s_r) \coloneqq \tau_0(s_r), \tag{4.98}$$

$$f'_1(s_r) = \begin{cases} 01 & \text{if } r = 1, \\ 1^{r-1}0 & \text{if } 2 \leq r \leq \sigma - 1, \\ 1^{\sigma-1} & \text{if } r = \sigma, \end{cases} \quad \tau'_1(s_r) = 0 \tag{4.99}$$

for $s_r \in \mathcal{S}$, where we suppose $\mathcal{S} = \{s_1, s_2, \ldots, s_\sigma\}$ and the notation 1^l denotes the sequence obtained by concatenating l copies of 1 for an integer $l \geq 1$. We can show that F' satisfies Definition 4.2.1 (i)–(vii) in a similar way to the case $\mathcal{R}_F = \{0, 1\}$.

□

4.3 Proofs of Lemmas in Chapter 4

4.3.1 Proof of Lemma 4.2.1

To prove Lemma 4.2.1, we first show the following Lemma 4.3.1.

Lemma 4.3.1. *For any $F \in \mathscr{F}_{\mathrm{AIFV}}$, the following conditions (i)–(iii) hold.*

(i) $\mathcal{P}^1_{F,0} = \mathcal{P}^1_{F,1} = \{0, 1\}$.

(ii) For any $i \in [F]$ and $b \in \mathcal{C}$, if $\mathcal{S}_{F,i}(\lambda) = \emptyset$ and $(i, b) \neq (1, 0)$, then $\mathcal{P}^1_{F,i}(b) = \{0, 1\}$.

(iii) For any $i \in [F]$ and $s \in \mathcal{S}$, if $\bar{\mathcal{P}}^0_{F,i}(f_i(s)) \neq \emptyset$, then $\bar{\mathcal{P}}^2_{F,i}(f_i(s)) = \{00\}$.

Proof of Lemma 4.3.1. (Proof of (i)): We first show

$$\mathcal{P}^1_{F,1} = \{0, 1\}. \tag{4.100}$$

To prove it, it suffices to show $|\bar{\mathcal{P}}^1_{F,1}| = 2$ because this implies $\mathcal{P}^1_{F,1} \supseteq \bar{\mathcal{P}}^1_{F,1} = \{0, 1\}$ by Lemma 2.2.1 (i).

- We obtain $|\bar{\mathcal{P}}^1_{F,1}| \neq 0$ by applying Corollary 2.3.1 (ii) (a) because $|\bar{\mathcal{P}}^0_{F,1}| \neq 0$ by Definition 4.2.1 (i) and Lemma 2.2.2 (iii).
- Also, we have $|\bar{\mathcal{P}}^1_{F,1}| \neq 1$ because neither the condition (a) nor (b) of Definition 4.2.1 (vii) holds for $(i, \boldsymbol{b}) = (1, \lambda)$ by Definition 4.2.1 (v).

These show (4.100).

Next, we show $\mathcal{P}^1_{F,0} = \{0, 1\}$ by considering the following two cases separately: the case $\mathcal{S}_{F,0}(\lambda) = \emptyset$ and the case $\mathcal{S}_{F,0}(\lambda) \neq \emptyset$.

- The case $\mathcal{S}_{F,0}(\lambda) = \emptyset$: By a similar argument to derive (4.100).

- The case $\mathcal{S}_{F,0}(\lambda) \neq \emptyset$: We have

$$\mathcal{P}^1_{F,0} \overset{\text{(A)}}{\supseteq} \bigcup_{s \in \mathcal{S}_{F,0}(\lambda)} \mathcal{P}^1_{F,\tau_0(s)} \overset{\text{(B)}}{=} \bigcup_{s \in \mathcal{S}_{F,0}(\lambda)} \mathcal{P}^1_{F,1} \overset{\text{(C)}}{=} \bigcup_{s \in \mathcal{S}_{F,0}(\lambda)} \{0,1\} \overset{\text{(D)}}{=} \{0,1\}, \tag{4.101}$$

where (A) follows from Lemma 2.2.1 (i), (B) follows from Definition 4.2.1 (iv) because $\bar{\mathcal{P}}^0_{F,0}(f_0(s)) = \bar{\mathcal{P}}^0_{F,0} \neq \emptyset$ by Definition 4.2.1 (i) and Lemma 2.2.2 (iii), (C) follows from (4.100), and (D) follows from $\mathcal{S}_{F,0}(\lambda) \neq \emptyset$.

(Proof of (ii)): Assume $\mathcal{S}_{F,i}(\lambda) = \emptyset$ and $(i,b) \neq (1,0)$. We consider the following two cases separately: the case $\mathcal{S}_{F,i}(b) = \emptyset$ and the case $\mathcal{S}_{F,i}(b) \neq \emptyset$.

- The case $\mathcal{S}_{F,i}(b) = \emptyset$: It suffices to show $|\bar{\mathcal{P}}^1_{F,1}(b)| = 2$ because this implies $\mathcal{P}^1_{F,i}(b) \supseteq \bar{\mathcal{P}}^1_{F,i}(b) = \{0,1\}$ by Lemma 2.2.1 (i).
  - We have $b \in \{0,1\} = \mathcal{P}^1_{F,i}$ by (i) of this lemma. Hence, there exists $\boldsymbol{x} \in \mathcal{S}^+$ such that $f^*_i(\boldsymbol{x}) \succeq b$. Since $\mathcal{S}_{F,i}(\lambda) = \mathcal{S}_{F,i}(b) = \emptyset$, we have $f_i(x_1) \succ b$ and thus $|\bar{\mathcal{P}}^1_{F,i}(b)| \neq 0$.
  - Also, by Definition 4.2.1 (vii), it must hold that $|\bar{\mathcal{P}}^1_{F,i}(b)| \neq 1$ since $\mathcal{S}_{F,i}(\lambda) = \mathcal{S}_{F,i}(b) = \emptyset$ and $(i,b) \neq (1,0)$.

  These show $\mathcal{P}^1_{F,i}(b) = \{0,1\}$ as desired.

- The case $\mathcal{S}_{F,i}(b) \neq \emptyset$: We have

$$\mathcal{P}^1_{F,i}(b) \overset{\text{(A)}}{\supseteq} \bigcup_{s \in \mathcal{S}_{F,i}(b)} \mathcal{P}^1_{F,\tau_i(s)} \overset{\text{(B)}}{=} \bigcup_{s \in \mathcal{S}_{F,i}(b)} \{0,1\} \overset{\text{(C)}}{=} \{0,1\} \tag{4.102}$$

  as desired, where (A) follows from Lemma 2.2.1 (i), (B) follows from (i) of this lemma, and (C) follows from $\mathcal{S}_{F,i}(b) \neq \emptyset$.

(Proof of (iii)): Assume $\bar{\mathcal{P}}^0_{F,i}(f_i(s)) \neq \emptyset$. Then we have $\bar{\mathcal{P}}^1_{F,i}(f_i(s)) \neq \emptyset$ by Corollary 2.3.1 (ii) (a). Since $1 \notin \bar{\mathcal{P}}^1_{F,i}(f_i(s))$ by Definition 4.2.1 (ii), it must hold that

$$\bar{\mathcal{P}}^1_{F,i}(f_i(s)) = \{0\}. \tag{4.103}$$

We have

$$0\mathcal{P}^1_{F,i}(f_i(s)0) \cup 1\mathcal{P}^1_{F,i}(f_i(s)1) \overset{\text{(A)}}{=} \bar{\mathcal{P}}^2_{F,i}(f_i(s)) \overset{\text{(B)}}{\subseteq} \{00,01\}, \tag{4.104}$$

where (A) follows from Lemma 2.2.1 (ii), and (B) follows from (4.103) and Lemma 2.3.2 (ii). Comparing both sides of (4.104), we have

$$1\mathcal{P}^1_{F,i}(f_i(s)1) = \emptyset. \tag{4.105}$$

Thus, we obtain

$$\begin{aligned}
\bar{\mathcal{P}}^2_{F,i}(f_i(s)) &\overset{\text{(A)}}{=} 0\mathcal{P}^1_{F,i}(f_i(s)0) \cup 1\mathcal{P}^1_{F,i}(f_i(s)1) && (4.106)\\
&\overset{\text{(B)}}{=} 0\mathcal{P}^1_{F,i}(f_i(s)0) && (4.107)\\
&\overset{\text{(C)}}{=} 0\Big(\bar{\mathcal{P}}^1_{F,i}(f_i(s)0) \cup \Big(\bigcup_{s' \in \mathcal{S}_{F,i}(f_i(s)0)} \mathcal{P}^1_{F,\tau_i(s')}\Big)\Big) && (4.108)\\
&= 0\bar{\mathcal{P}}^1_{F,i}(f_i(s)0) \cup \Big(\bigcup_{s' \in \mathcal{S}_{F,i}(f_i(s)0)} 0\mathcal{P}^1_{F,\tau_i(s')}\Big) && (4.109)\\
&\overset{\text{(D)}}{=} 0\bar{\mathcal{P}}^1_{F,i}(f_i(s)0) && (4.110)\\
&\overset{\text{(E)}}{=} \{00\}, && (4.111)
\end{aligned}$$

where (A) follows from Lemma 2.2.1 (ii), (B) follows from (4.105), (C) follows from Lemma 2.2.1 (i), (D) follows since $\mathcal{S}_{F,i}(f_i(s)0) = \emptyset$ by Definition 4.2.1 (iii), and (E) follows from (4.103).

□

Proof of Lemma 4.2.1. We fix $F \in \mathscr{F}_{\text{AIFV}}$ arbitrarily and show $F \in \mathscr{F}_{\text{reg}}$, $F \in \mathscr{F}_{\text{2-dec}}$, $\mathcal{P}^2_{F,0} = \{00, 01, 10, 11\}$ and $\mathcal{P}^2_{F,1} = \{01, 10, 11\}$.

(Proof of $F \in \mathscr{F}_{\text{reg}}$): By Lemma 2.2.2 (ii), the following (4.112) holds, which implies

$$\begin{aligned}
&{}^\forall i \in [F]; {}^\exists s \in \mathcal{S}; \bar{\mathcal{P}}^0_{F,i}(f_i(s)) = \emptyset && (4.112)\\
&\overset{\text{(A)}}{\Longrightarrow} {}^\forall i \in [F]; {}^\exists s \in \mathcal{S}; \tau_i(s) = 0 && (4.113)\\
&\overset{\text{(B)}}{\Longrightarrow} \mathcal{R}_F \ni 0 && (4.114)\\
&\overset{\text{(C)}}{\Longrightarrow} F \in \mathscr{F}_{\text{reg}}, && (4.115)
\end{aligned}$$

where (A) follows from Definition 4.2.1 (iv), (B) follows from (2.68), and (C) follows from Lemma 2.5.2 (i).

(Proof of $\mathcal{P}^2_{F,1} = \{01, 10, 11\}$): We have $0 \in \{0, 1\} = \mathcal{P}^1_{F,1}$ by Lemma 4.3.1 (i). Hence, there exists $\boldsymbol{x} \in \mathcal{S}^+$ such that $f^*_1(\boldsymbol{x}) \succeq 0$. By Definition 4.2.1 (v), we have $f_1(x_1) \succ 0$ and thus

$$\bar{\mathcal{P}}^1_{F,1}(0) \neq \emptyset. \tag{4.116}$$

Therefore, we obtain

$$\mathcal{P}^1_{F,1}(0) \overset{\text{(A)}}{=} \bar{\mathcal{P}}^1_{F,1}(0) \cup \Big(\bigcup_{s' \in \mathcal{S}_{F,1}(0)} \mathcal{P}^1_{F,\tau_1(s')} \Big) \overset{\text{(B)}}{=} \bar{\mathcal{P}}^1_{F,1}(0) \overset{\text{(C)}}{=} \{1\}, \tag{4.117}$$

where (A) follows from Lemma 2.2.1 (i), (B) follows since $\mathcal{S}_{F,1}(0) = \emptyset$ by Definition 4.2.1 (v), and (C) follows from (4.116) and Definition 4.2.1 (vi). Thus, we obtain

$$\mathcal{P}^2_{F,1} \overset{\text{(A)}}{=} \bar{\mathcal{P}}^2_{F,1} \cup \Big(\bigcup_{s' \in \mathcal{S}_{F,1}(\lambda)} \mathcal{P}^2_{F,\tau_1(s')} \Big) \tag{4.118}$$

$$\overset{\text{(B)}}{=} \bar{\mathcal{P}}^2_{F,1} \tag{4.119}$$

$$\overset{\text{(C)}}{=} 0\mathcal{P}^1_{F,1}(0) \cup 1\mathcal{P}^1_{F,1}(1) \tag{4.120}$$

$$\overset{\text{(D)}}{=} 0\{1\} \cup 1\mathcal{P}^1_{F,1}(1) \tag{4.121}$$

$$\overset{\text{(E)}}{=} 0\{1\} \cup 1\{0,1\} \tag{4.122}$$

$$= \{01, 10, 11\} \tag{4.123}$$

as desired, where (A) follows from Lemma 2.2.1 (i), (B) follows since $\mathcal{S}_{F,1}(\lambda) = \emptyset$ by Definition 4.2.1 (v), (C) follows from Lemma 2.2.1 (ii), (D) follows from (4.117), and (E) follows from Lemma 4.3.1 (ii) since $\mathcal{S}_{F,1}(\lambda) = \emptyset$ by Definition 4.2.1 (v).

(Proof of $\mathcal{P}^2_{F,0} = \{00, 01, 10, 11\}$): We consider the following two cases separately: the case $\mathcal{S}_{F,0}(\lambda) = \emptyset$ and the case $\mathcal{S}_{F,0}(\lambda) \neq \emptyset$.

- The case $\mathcal{S}_{F,0}(\lambda) = \emptyset$: We have

$$\mathcal{P}^2_{F,0} \overset{\text{(A)}}{\supseteq} \bar{\mathcal{P}}^2_{F,0} \tag{4.124}$$

$$\overset{\text{(B)}}{=} 0\mathcal{P}^1_{F,0}(0) \cup 1\mathcal{P}^1_{F,1}(1) \tag{4.125}$$

$$\overset{\text{(C)}}{=} 0\{0,1\} \cup 1\mathcal{P}^1_{F,1}(1) \tag{4.126}$$

$$\overset{\text{(D)}}{=} 0\{0,1\} \cup 1\{0,1\} \tag{4.127}$$

$$= \{00, 01, 10, 11\} \tag{4.128}$$

as desired, where (A) follows from Lemma 2.2.1 (i), (B) follows from Lemma 2.2.1 (ii), (C) follows from Lemma 4.3.1 (ii) since $\mathcal{S}_{F,0}(\lambda) = \emptyset$, and (D) follows from Lemma 4.3.1 (ii) since $\mathcal{S}_{F,1}(\lambda) = \emptyset$ by Definition 4.2.1 (v).

- The case $\mathcal{S}_{F,0}(\lambda) \neq \emptyset$: Let $s \in \mathcal{S}_{F,0}(\lambda) \neq \emptyset$. We have

$$\bar{\mathcal{P}}^0_{F,0}(f_0(s)) = \bar{\mathcal{P}}^0_{F,0} \neq \emptyset \tag{4.129}$$

by Definition 4.2.1 (i) and Lemma 2.2.2 (iii), and thus we have $\tau_0(s) = 1$ by Definition 4.2.1 (iv). Hence, we have

$$\begin{aligned}
\mathcal{P}^2_{F,0} &\overset{\text{(A)}}{=} \bar{\mathcal{P}}^2_{F,0} \cup \Big(\bigcup_{s' \in \mathcal{S}_{F,0}(\lambda)} \mathcal{P}^2_{F,\tau_0(s')} \Big) && (4.130)\\
&\overset{\text{(B)}}{\supseteq} \bar{\mathcal{P}}^2_{F,0}(f_0(s)) \cup \mathcal{P}^2_{F,\tau_0(s)} && (4.131)\\
&\overset{\text{(C)}}{=} \bar{\mathcal{P}}^2_{F,0}(f_0(s)) \cup \mathcal{P}^2_{F,1} && (4.132)\\
&\overset{\text{(D)}}{=} \{00\} \cup \mathcal{P}^2_{F,1} && (4.133)\\
&\overset{\text{(E)}}{=} \{00\} \cup \{01, 10, 11\} && (4.134)\\
&= \{00, 01, 10, 11\} && (4.135)
\end{aligned}$$

as desired, where (A) follows from Lemma 2.2.1 (i), (B) follows from $s \in \mathcal{S}_{F,0}(\lambda)$, (C) follows from $\tau_0(s) = 1$, (D) follows from (4.129) and Lemma 4.3.1 (iii), and (E) follows from (4.123).

(Proof of $F \in \mathscr{F}_{2\text{-dec}}$): Since f_0 and f_1 are injective by Definition 4.2.1 (i), the code-tuple F satisfies Definition 2.2.3 (b) (cf. Remark 2.2.1). We show that F satisfies Definition 2.2.3 (a). We choose $i \in [2]$ and $s \in \mathcal{S}$ arbitrarily and show $\mathcal{P}^2_{F,\tau_i(s)} \cap \bar{\mathcal{P}}^2_{F,i}(f_i(s)) = \emptyset$ for the following two cases: the case $\bar{\mathcal{P}}^0_{F,i}(f_i(s)) = \emptyset$ and the case $\bar{\mathcal{P}}^0_{F,i}(f_i(s)) \neq \emptyset$.

- The case $\bar{\mathcal{P}}^0_{F,i}(f_i(s)) = \emptyset$: We have

$$\mathcal{P}^2_{F,\tau_i(s)} \cap \bar{\mathcal{P}}^2_{F,i}(f_i(s)) \overset{\text{(A)}}{=} \mathcal{P}^2_{F,\tau_i(s)} \cap \emptyset = \emptyset \tag{4.136}$$

as desired, where (A) follows from $\bar{\mathcal{P}}^0_{F,i}(f_i(s)) = \emptyset$ and Corollary 2.3.1 (ii) (a).

- The case $\bar{\mathcal{P}}^0_{F,i}(f_i(s)) \neq \emptyset$: We have

$$\begin{aligned}
\mathcal{P}^2_{F,\tau_i(s)} \cap \bar{\mathcal{P}}^2_{F,i}(f_i(s)) &\overset{(A)}{=} \mathcal{P}^2_{F,1} \cap \bar{\mathcal{P}}^2_{F,i}(f_i(s)) && (4.137)\\
&\overset{(B)}{=} \{01, 10, 11\} \cap \bar{\mathcal{P}}^2_{F,i}(f_i(s)) && (4.138)\\
&\overset{(C)}{=} \{01, 10, 11\} \cap \{00\} && (4.139)\\
&= \emptyset && (4.140)
\end{aligned}$$

as desired, where (A) follows from $\bar{\mathcal{P}}^0_{F,i}(f_i(s)) \neq \emptyset$ and Definition 2.2.3 (iv), (B) follows from (4.123), and (C) follows from $\bar{\mathcal{P}}^0_{F,i}(f_i(s)) \neq \emptyset$ and Lemma 4.3.1 (iii).

□

4.3.2 Proof of Lemma 4.2.4

Proof of Lemma 4.2.4. (Proof of (i)): We have $\mathcal{P}^1_{F,i} = \{0, 1\}$ by $F \in \mathscr{F}_1$. Hence, by Lemma 2.3.2 (i), there exist $a, b \in \mathcal{C}$ such that $0a, 1b \in \mathcal{P}^2_{F,i}$.

(Proof of (ii) (a)): Assume $|\mathcal{P}^2_{F,i}| = 2$. We prove by contradiction assuming that $|f_i(s)| \leq 1$ for some $s \in \mathcal{S}$. We consider the following two cases separately: the case $|f_i(s)| = 0$ and the case $|f_i(s)| = 1$.

- The case $|f_i(s)| = 0$: We have

$$\begin{aligned}
|\bar{\mathcal{P}}^0_{F,i}| + 2|\mathcal{S}_{F,i}(\lambda)| &\overset{(A)}{\leq} |\bar{\mathcal{P}}^2_{F,i}| + 2|\mathcal{S}_{F,i}(\lambda)| && (4.141)\\
&\overset{(B)}{\leq} |\bar{\mathcal{P}}^2_{F,i}| + \sum_{s' \in \mathcal{S}_{F,i}(\lambda)} |\mathcal{P}^2_{F,\tau_i(s')}| && (4.142)\\
&\overset{(C)}{=} |\mathcal{P}^2_{F,i}| && (4.143)\\
&\overset{(D)}{=} 2, && (4.144)
\end{aligned}$$

where (A) follows from Corollary 2.3.1 (ii) (b), (B) follows since $|\mathcal{P}^2_{F,\tau_i(s')}| \geq 2$ for any $s' \in \mathcal{S}_{F,i}(\lambda)$ by (i) of this lemma, (C) follows from Lemma 2.2.3, and (D) follows directly from the assumption.

Also, by $|f_i(s)| = 0$, we have

$$|\mathcal{S}_{F,i}(\lambda)| \geq |\{s\}| = 1. \quad (4.145)$$

By (4.141) and (4.145), we have

$$|\bar{\mathcal{P}}^0_{F,i}| = 0 \tag{4.146}$$

and

$$|\mathcal{S}_{F,i}(\lambda)| = 1. \tag{4.147}$$

By (4.147) and Lemma 2.2.2 (iii), we obtain $\bar{\mathcal{P}}^0_{F,i} \neq \emptyset$, which conflicts with (4.146).

- The case $|f_i(s)| = 1$: Put $f_i(s) = c \in \mathcal{C}$. We have

$$\mathcal{P}^2_{F,i} \overset{\text{(A)}}{\supseteq} \bar{\mathcal{P}}^2_{F,i} \overset{\text{(B)}}{\supseteq} c\mathcal{P}^1_{F,i} \overset{\text{(C)}}{=} c\{0,1\} = \{c0, c1\}, \tag{4.148}$$

where (A) follows from Lemma 2.2.1 (i), (B) follows from Lemma 2.2.1 (ii), and (C) follows from $F \in \mathscr{F}_1$. Also, by (i) of this lemma, we have

$$\mathcal{P}^2_{F,i} \supseteq \{ca, \bar{c}b\} \tag{4.149}$$

for some $a, b \in \mathcal{C}$. By (4.148) and (4.149), we have $|\mathcal{P}^2_{F,i}| \geq |\{c0, c1, \bar{c}b\}| = 3$, which conflicts with $|\mathcal{P}^2_{F,i}| = 2$.

(Proof of (ii) (b)): Assume $|\mathcal{P}^2_{F,i}| = 2$. We have

$$\bar{\mathcal{P}}^2_{F,i} \overset{\text{(A)}}{=} \bar{\mathcal{P}}^2_{F,i} \cup \Big(\bigcup_{s \in \mathcal{S}_{F,i}(\lambda)} \mathcal{P}^k_{F,\tau_i(s)} \Big) \overset{\text{(B)}}{=} \mathcal{P}^2_{F,i} \overset{\text{(C)}}{=} \{0a, 1b\} \tag{4.150}$$

for some $a, b \in \mathcal{C}$ as desired, where (A) follows because $\mathcal{S}_{F,i}(\lambda) = \emptyset$ by (ii) (a) of this lemma, (B) follows from Lemma 2.2.1 (i), and (C) follows from (i) of this lemma and $|\mathcal{P}^2_{F,i}| = 2$.

(Proof of (iii)): Assume $s \neq s'$ and $f_i(s) = f_i(s')$. We have

$$\begin{aligned}
&|\bar{\mathcal{P}}^2_{F,i}(f_i(s))| + |\mathcal{P}^2_{F,\tau_i(s)}| + |\mathcal{P}^2_{F,\tau_i(s')}| \\
&\overset{\text{(A)}}{\leq} |\bar{\mathcal{P}}^2_{F,i}(f_i(s))| + \sum_{s'' \in \mathcal{S}_{F,i}(f_i(s))} |\mathcal{P}^2_{F,\tau_i(s'')}| && (4.151) \\
&\overset{\text{(B)}}{=} |\mathcal{P}^2_{F,i}(f_i(s))| && (4.152) \\
&\leq 4, && (4.153)
\end{aligned}$$

where (A) follows from $s \neq s'$ and $f_i(s) = f_i(s')$, and (B) follows from Lemma 2.2.3.

Also, by (i) of this lemma, we have

$$|\mathcal{P}^2_{F,\tau_i(s)}| \geq 2, \quad |\mathcal{P}^2_{F,\tau_i(s')}| \geq 2. \tag{4.154}$$

By (4.151) and (4.154), it must hold that $|\bar{\mathcal{P}}^2_{F,i}(f_i(s))| = 0$ and $|\mathcal{P}^2_{F,\tau_i(s)}| = |\mathcal{P}^2_{F,\tau_i(s')}| = 2$ as desired.

(Proof of (iv)): We have

$$\begin{aligned}
|\mathcal{S}_{F,i}(f_i(s))| &= \frac{2|\mathcal{S}_{F,i}(f_i(s))|}{2} && (4.155)\\
&\overset{\text{(A)}}{\leq} \frac{\sum_{s' \in \mathcal{S}_{F,i}(f_i(s))} |\mathcal{P}^2_{F,\tau_i(s')}|}{2} && (4.156)\\
&\overset{\text{(B)}}{=} \frac{|\mathcal{P}^2_{F,i}(f_i(s))| - |\bar{\mathcal{P}}^2_{F,i}(f_i(s))|}{2} && (4.157)\\
&\leq \frac{4 - |\bar{\mathcal{P}}^2_{F,i}(f_i(s))|}{2} && (4.158)\\
&\overset{\text{(C)}}{\leq} \frac{4 - |\bar{\mathcal{P}}^0_{F,i}(f_i(s))|}{2} && (4.159)\\
&\leq \begin{cases} \frac{3}{2} & \text{if } \bar{\mathcal{P}}^0_{F,i}(f_i(s)) \neq \emptyset, \\ 2 & \text{if } \bar{\mathcal{P}}^0_{F,i}(f_i(s)) = \emptyset, \end{cases} && (4.160)
\end{aligned}$$

as desired, where (A) follows since $|\mathcal{P}^2_{F,\tau_i(s')}| \geq 2$ for any $s' \in \mathcal{S}_{F,i}(f_i(s))$ by (i) of this lemma, (B) follows from Lemma 2.2.3, and (C) follows from Corollary 2.3.1 (ii) (b).

(Proof of (v)): We prove by contradiction assuming that there exist $s, s' \in \mathcal{S}$ and $c \in \mathcal{C}$ such that

$$f_i(s') = f_i(s)c. \tag{4.161}$$

By (i) of this lemma, we have

$$\mathcal{P}^2_{F,\tau_i(s)} \ni cc' \tag{4.162}$$

for some $c' \in \mathcal{C}$. Also, we have

$$\bar{\mathcal{P}}^2_{F,i}(f_i(s)) \overset{\text{(A)}}{\supseteq} c\mathcal{P}^1_{F,i}(f_i(s)c) \overset{\text{(B)}}{=} c\mathcal{P}^1_{F,i}(f_i(s')) \overset{\text{(C)}}{\supseteq} c\mathcal{P}^1_{F,\tau_i(s')} \overset{\text{(D)}}{=} c\{0,1\} \ni cc', \tag{4.163}$$

where (A) follows from Lemma 2.2.1 (ii), (B) follows from (4.161), (C) follows from Lemma 2.2.1 (i), and (D) follows from $F \in \mathscr{F}_1$. By (4.162) and (4.163), we obtain $\mathcal{P}^2_{F,\tau_i(s)} \cap \bar{\mathcal{P}}^2_{F,i}(f_i(s)) \neq \emptyset$, which conflicts with $F \in \mathscr{F}_{2\text{-dec}}$.

(Proof of (vi)): We prove by contradiction assuming that there exist $s \in \mathcal{S}$ and $c \in \mathcal{C}$ such that

$$\bar{\mathcal{P}}^1_{F,i}(f_i(s)c) = \{0,1\}. \tag{4.164}$$

By (i) of this lemma, we have

$$\mathcal{P}^2_{F,\tau_i(s)} \ni cc' \tag{4.165}$$

for some $c' \in \mathcal{C}$. Also, we have

$$\bar{\mathcal{P}}^2_{F,i}(f_i(s)) \overset{\text{(A)}}{\supseteq} c\mathcal{P}^1_{F,i}(f_i(s)c) \overset{\text{(B)}}{\supseteq} c\bar{\mathcal{P}}^1_{F,i}(f_i(s)c) \overset{\text{(C)}}{=} c\{0,1\} \ni cc', \tag{4.166}$$

where (A) follows from Lemma 2.2.1 (ii), (B) follows from Lemma 2.2.1 (i), and (C) follows from (4.164). By (4.165) and (4.166), we obtain $\mathcal{P}^2_{F,\tau_i(s)} \cap \bar{\mathcal{P}}^2_{F,i}(f_i(s)) \neq \emptyset$, which conflicts with $F \in \mathscr{F}_{\text{2-dec}}$. □

4.3.3 Proof of Lemma 4.2.5

To prove Lemma 4.2.5, we use the following Lemma 4.3.2 obtained by Theorem 3.1.2 with $k = 2$.

Lemma 4.3.2. *For any $F \in \mathscr{F}_{\text{2-opt}}$, $i \in \mathcal{R}_F$, and $\boldsymbol{b} = b_1 b_2 \ldots b_l \in \mathcal{C}^{\geq 2}$, if $b_1 b_2 \in \mathcal{P}^2_{F,i}$, then $\boldsymbol{b} \in \mathcal{P}^*_{F,i}$.*

Proof of Lemma 4.2.5. Assume $\bar{\mathcal{P}}^0_{F,i}(f_i(s)) = \emptyset$ and $|\mathcal{S}_{F,i}(f_i(s))| = 1$. We prove by contradiction assuming $|\mathcal{P}^2_{F,\tau_i(s)}| < 4$, that is, there exists

$$\boldsymbol{b} = b_1 b_2 \in \mathcal{C}^2 \setminus \mathcal{P}^2_{F,\tau_i(s)}. \tag{4.167}$$

First, we put

$$\boldsymbol{d} = d_1 d_2 \ldots d_l \coloneqq f_i(s)\boldsymbol{b} \tag{4.168}$$

and show

$$d_1 d_2 \in \mathcal{P}^2_{F,i} \tag{4.169}$$

considering the following three cases separately: the case $|f_i(s)| = 0$, the case $|f_i(s)| = 1$, and the case $|f_i(s)| \geq 2$.

- The case $|f_i(s)| = 0$: We have

$$\bar{\mathcal{P}}^0_{F,i}(f_i(s)) \overset{\text{(A)}}{=} \bar{\mathcal{P}}^0_{F,i} \overset{\text{(B)}}{\neq} \emptyset, \tag{4.170}$$

  where (A) follows from $|f_i(s)| = 0$, and (B) follows from $|\mathcal{S}_{F,i}(f_i(s))| = 1$ and Lemma 2.2.2 (iii). This conflicts with the assumption. Therefore, the case $|f_i(s)| = 0$ is impossible.

- The case $|f_i(s)| = 1$: Then we have $f_i(s) = d_1$ by (4.168). Also, we have $d_2 \in \{0, 1\} = \mathcal{P}^1_{F,\tau_i(s)}$ by $F \in \mathscr{F}_1$. Thus, there exists $\boldsymbol{x} \in \mathcal{S}^+$ such that $f^*_{\tau_i(s)}(\boldsymbol{x}) \succeq d_2$. Then we have $f^*_i(s\boldsymbol{x}) = f_i(s) f^*_{\tau_i(s)}(\boldsymbol{x}) \succeq d_1 d_2$, which leads to (4.169).
- The case $|f_i(s)| \geq 2$: Directly from $f_i(s) \succeq d_1 d_2$ by (4.168).

Consequently, (4.169) holds.

By $i \in \mathcal{R}_F$ and (4.169), we obtain $\boldsymbol{d} \in \mathcal{P}^*_{F,i}$ applying Lemma 4.3.2. Hence, there exists $\boldsymbol{y} \in \mathcal{S}^+$ such that

$$f^*_i(\boldsymbol{y}) \succeq \boldsymbol{d}. \tag{4.171}$$

By (4.168) and (4.171), exactly one of $f_i(y_1) \succ f_i(s)$ and $f_i(y_1) \preceq f_i(s)$ holds. Now, the latter $f_i(y_1) \preceq f_i(s)$ must hold because the former $f_i(y_1) \succ f_i(s)$ conflicts with $\bar{\mathcal{P}}^0_{F,i}(f_i(s)) = \emptyset$ by Lemma 2.2.2 (i). Therefore, there exists $\boldsymbol{c} \in \mathcal{C}^*$ such that $f_i(y_1)\boldsymbol{c} = f_i(s)$. We divide into the following three cases by $|\boldsymbol{c}|$.

- The case $|\boldsymbol{c}| = 0$: We have $f_i(y_1) = f_i(s)$, which leads to $y_1 = s$ by $|\mathcal{S}_{F,i}(f_i(s))| = 1$. Hence, we have

  $$f_i(s) f^*_{\tau_i(s)}(\mathrm{suff}(\boldsymbol{y})) = f_i(y_1) f^*_{\tau_i(y_1)}(\mathrm{suff}(\boldsymbol{y})) = f^*_i(\boldsymbol{y}) \overset{\text{(A)}}{\succeq} \boldsymbol{d} \overset{\text{(B)}}{=} f_i(s)\boldsymbol{b}, \tag{4.172}$$

  where (A) follows from (4.171), and (B) follows from (4.168). Comparing both sides, we obtain $f^*_{\tau_i(s)}(\mathrm{suff}(\boldsymbol{y})) \succeq \boldsymbol{b}$. This leads to $\boldsymbol{b} \in \mathcal{P}^2_{F,\tau_i(s)}$, which conflicts with (4.167).
- The case $|\boldsymbol{c}| = 1$: We have $f_i(y_1) = f_i(s)c_1$, which conflicts with Lemma 4.2.4 (v).
- The case $|\boldsymbol{c}| \geq 2$: We have

  $$f_i(y_1)c_1c_2 \preceq f_i(s), \tag{4.173}$$

  which leads to

  $$c_1c_2 \in \bar{\mathcal{P}}^2_{F,i}(f_i(y_1)). \tag{4.174}$$

  Also, we have

  $$f_i(y_1) f^*_{\tau_i(y_1)}(\mathrm{suff}(\boldsymbol{y})) = f^*_i(\boldsymbol{y}) \overset{\text{(A)}}{\succeq} \boldsymbol{d} \overset{\text{(B)}}{=} f_i(s)\boldsymbol{b} \succeq f_i(s) \overset{\text{(C)}}{\succeq} f_i(y_1)c_1c_2, \tag{4.175}$$

where (A) follows from (4.171), (B) follows from (4.168), and (C) follows from (4.173). Comparing both sides, we obtain $f^*_{\tau_i(y_1)}(\mathrm{suff}(\boldsymbol{y})) \succeq c_1c_2$, which leads to

$$c_1c_2 \in \mathcal{P}^2_{F,\tau_i(y_1)}. \tag{4.176}$$

By (4.174) and (4.176), we obtain $\bar{\mathcal{P}}^2_{F,i}(f_i(y_1)) \cap \mathcal{P}^2_{F,\tau_i(y_1)} \neq \emptyset$, which conflicts with $F \in \mathscr{F}_{2\text{-dec}}$.

□

4.3.4 Proof of Lemma 4.2.7 (iii)

To prove Lemma 4.2.7 (iii), we prove the following Lemmas 4.3.3 and 4.3.4.

Lemma 4.3.3. *Let $F \in \mathscr{F}_1$, $i \in [F]$, and $s, s' \in \mathcal{S}$, and let $\gamma(s_1)\gamma(s_2)\ldots\gamma(s_\rho)$ (resp. $\gamma(s'_1)\gamma(s'_2)\ldots\gamma(s'_{\rho'})$) be the γ-decomposition of $f_i(s)$ (resp. $f_i(s')$). For any $r = 1, 2, \ldots, m \coloneqq \min\{\rho, \rho'\}$, if one of the following conditions (a) and (b) holds, then $\gamma(s_r) = \gamma(s'_r) \iff \dot{\gamma}(s_r) = \dot{\gamma}(s'_r)$.*

(a) $r = 1$.

(b) $r \geq 2$ *and* $s_{r-1} = s'_{r-1}$.

Proof of Lemma 4.3.3. Assume that the condition (a) or (b) holds.

($\implies$) Directly from (4.24).

($\impliedby$) We prove the contraposition. Namely, we prove $\dot{\gamma}(s_r) \neq \dot{\gamma}(s'_r)$ assuming $\gamma(s_r) \neq \gamma(s'_r)$. Put $\gamma(s_r) = g_1g_2\ldots g_l$ and $\gamma(s'_r) = g'_1g'_2\ldots g'_{l'}$. We consider the following two cases separately: the case $|\gamma(s_r)| \neq |\gamma(s'_r)|$ and the case $|\gamma(s_r)| = |\gamma(s'_r)|$.

- The case $|\gamma(s_r)| \neq |\gamma(s'_r)|$: We have

$$|\dot{\gamma}(s_r)| \overset{\text{(A)}}{=} |\gamma(s_r)| \overset{\text{(B)}}{\neq} |\gamma(s'_r)| \overset{\text{(C)}}{=} |\dot{\gamma}(s'_r)|, \tag{4.177}$$

  where (A) follows from Lemma 4.2.7 (i), (B) follows from the assumption, and (C) follows from Lemma 4.2.7 (i). This implies $\dot{\gamma}(s_r) \neq \dot{\gamma}(s'_r)$ as desired.

- The case $|\gamma(s_r)| = |\gamma(s'_r)|$: If $|\gamma(s_r)| = |\gamma(s'_r)| \geq 3$ and $g_3 g_4 \dots g_l \neq g'_3 g'_4 \dots g'_{l'}$, then we obtain $\dot{\gamma}(s_r) \neq \dot{\gamma}(s'_r)$ directly from (4.24). Thus, we assume
$$g_j \neq g'_j \text{ for some } 1 \leq j \leq \min\{2, |\gamma(s_r)|\}. \tag{4.178}$$
We divide into the following two cases by which of the conditions (a) and (b) holds: the case $r = 1$ and the case $r \geq 2, s_{r-1} = s'_{r-1}$.

  - The case $r = 1$: We consider the following two cases separately: the case $|\mathcal{P}^2_{F,i}| = 2$ and the case $|\mathcal{P}^2_{F,i}| \geq 3$.
    * The case $|\mathcal{P}^2_{F,i}| = 2$: By Lemma 4.2.4 (ii), we have $\mathcal{P}^2_{F,i} = \{0a, 1b\}$ for some $a, b \in \mathcal{C}$ and we have $|\gamma(s_1)| = |\gamma(s'_1)| \geq 2$. This shows $g_1 g_2, g'_1 g'_2 \in \{0a, 1b\}$. Hence, since $g_1 g_2 \neq g'_1 g'_2$ by (4.178), we may assume
    $$g_1 \neq g'_1. \tag{4.179}$$
    Thus, we obtain
    $$\dot{\gamma}(s_r) \overset{\text{(A)}}{=} a_{F,i} g_1 g_3 g_4 \dots g_l \overset{\text{(B)}}{\neq} a_{F,i} g'_1 g'_3 g'_4 \dots g'_{l'} \overset{\text{(C)}}{=} \dot{\gamma}(s'_r) \tag{4.180}$$
    as desired, where (A) follows from the first case of (4.24) since $r = 1$ and $|\mathcal{P}^2_{F,i}| = 2$, (B) follows from (4.179), and (C) follows from the first case of (4.24) since $r = 1$ and $|\mathcal{P}^2_{F,i}| = 2$.
    * The case $|\mathcal{P}^2_{F,i}| \geq 3$: We obtain
    $$\dot{\gamma}(s_r) \overset{\text{(A)}}{=} \gamma(s_r) \overset{\text{(B)}}{\neq} \gamma(s'_r) \overset{\text{(C)}}{=} \dot{\gamma}(s'_r) \tag{4.181}$$
    as desired, where (A) follows from the second case of (4.24) since $r = 1$ and $|\mathcal{P}^2_{F,i}| \geq 3$, (B) follows from (4.178), and (C) follows from the second case of (4.24) since $r = 1$ and $|\mathcal{P}^2_{F,i}| \geq 3$.
  - The case $r \geq 2, s_{r-1} = s'_{r-1}$: By Lemma 4.2.6 (iii), we have $g_1 g_2 \in \bar{\mathcal{P}}^2_{F,i}(f_i(s_{r-1}))$ and $g'_1 g'_2 \in \bar{\mathcal{P}}^2_{F,i}(f_i(s'_{r-1}))$. Since $s_{r-1} = s'_{r-1}$, we have
  $$\{g_1 g_2, g'_1 g'_2\} \subseteq \bar{\mathcal{P}}^2_{F,i}(f_i(s_{r-1})). \tag{4.182}$$
  Now, we show
  $$g_1 \neq g'_1 \tag{4.183}$$

by contradiction assuming the contrary $g_1 = g_1'$. Then by (4.178), it must hold that $|\gamma(s_r)| = |\gamma(s_r')| \geq 2$ and $g_2 \neq g_2'$. Hence, we have

$$\begin{aligned}
& g_1 \mathcal{P}^1_{F,i}(f_i(s_{r-1})g_1) \cup \bar{g}_1 \mathcal{P}^1_{F,i}(f_i(s_{r-1})\bar{g}_1) \\
& \stackrel{\text{(A)}}{=} \bar{\mathcal{P}}^2_{F,i}(f_i(s_{r-1})) && (4.184) \\
& \stackrel{\text{(B)}}{\supseteq} \{g_1 g_2, g_1' g_2'\} && (4.185) \\
& \stackrel{\text{(C)}}{=} g_1\{g_2, g_2'\} && (4.186) \\
& \stackrel{\text{(D)}}{=} g_1\{0, 1\}, && (4.187)
\end{aligned}$$

where (A) follows from Lemma 2.2.1 (ii), (B) follows from (4.182), (C) follows from $g_1 = g_1'$ and (4.183), and (D) follows from $g_2 \neq g_2'$. Comparing both sides of (4.184), we obtain $\mathcal{P}^1_{F,i}(f_i(s_{r-1})g_1) = \{0, 1\}$, which conflicts with Lemma 4.2.4 (vi). Hence, we conclude that (4.183) holds.

We have

$$|\bar{\mathcal{P}}^1_{F,i}(f_i(s_{r-1}))| \stackrel{\text{(A)}}{=} |\{g_1, g_1'\}| \stackrel{\text{(B)}}{=} |\{0, 1\}| = 2, \tag{4.188}$$

where (A) follows from (4.182) and Lemma 2.3.2 (ii), and (B) follows from (4.183). Therefore, we obtain

$$\dot{\gamma}(s_r) \stackrel{\text{(A)}}{=} \bar{a}_{F,\tau_i(s_{r-1})} g_1 g_3 g_4 \ldots g_l \stackrel{\text{(B)}}{\neq} \bar{a}_{F,\tau_i(s'_{r-1})} g_1' g_3' g_4' \ldots g_{l'}' \stackrel{\text{(C)}}{=} \dot{\gamma}(s_r') \tag{4.189}$$

as desired, where (A) follows from the third case of (4.24) since $r \geq 2$ and (4.188) hold, (B) follows from (4.183), and (C) follows from the third case of (4.24) since $r \geq 2$ and (4.188) hold.

□

Lemma 4.3.4. *Let $F \in \mathscr{F}_1$, $i \in [F]$, and $s, s' \in \mathcal{S}$, and let $\gamma(s_1)\gamma(s_2)\ldots\gamma(s_\rho)$ (resp. $\gamma(s_1')\gamma(s_2')\ldots\gamma(s_{\rho'}')$) be the γ-decomposition of $f_i(s)$ (resp. $f_i(s')$). If $\dot{f}_i(s) \preceq \dot{f}_i(s')$, then for any $r = 1, 2, \ldots, m \coloneqq \min\{\rho, \rho'\}$, we have $\gamma(s_r) = \gamma(s_r')$.*

Proof of Lemma 4.3.4. Assume

$$\dot{f}_i(s) \preceq \dot{f}_i(s'). \tag{4.190}$$

It suffices to prove that the following conditions (a) and (b) hold for any $r = 1, 2, \ldots, m$ by induction for r.

(a) $\gamma(s_r) = \gamma(s'_r)$.

(b) If $r \neq m$, then $s_r = s'_r$.

We fix $q \geq 1$ and show that (a) and (b) hold for $r = q$ under the assumption that (a) and (b) hold for any $r = 1, 2, \ldots, q-1$.

We first show that the condition (a) holds for $r = q$. We have

$$\begin{aligned}
&\dot{f}_i(s_{q-1})\dot{\gamma}(s_q)\dot{\gamma}(s_{q+1})\ldots\dot{\gamma}(s_\rho) \\
&\quad = \dot{f}_i(s) && (4.191)\\
&\quad \overset{\text{(A)}}{\preceq} \dot{f}_i(s') && (4.192)\\
&\quad = \dot{f}_i(s'_{q-1})\dot{\gamma}(s'_q)\dot{\gamma}(s'_{q+1})\ldots\dot{\gamma}(s'_{\rho'}) && (4.193)\\
&\quad \overset{\text{(B)}}{=} \dot{f}_i(s_{q-1})\dot{\gamma}(s'_q)\dot{\gamma}(s'_{q+1})\ldots\dot{\gamma}(s'_{\rho'}), && (4.194)
\end{aligned}$$

where we suppose $\dot{f}_i(s_{q-1}) \coloneqq \lambda$ for the case $q = 1$, and (A) follows from (4.190), and (B) follows from the induction hypothesis. Comparing both sides, we have

$$\dot{\gamma}(s_q)\dot{\gamma}(s_{q+1})\ldots\dot{\gamma}(s_\rho) \preceq \dot{\gamma}(s'_q)\dot{\gamma}(s'_{q+1})\ldots\dot{\gamma}(s'_{\rho'}). \qquad (4.195)$$

Hence, at least one of $\dot{\gamma}(s_q) \preceq \dot{\gamma}(s'_q)$ and $\dot{\gamma}(s_q) \succeq \dot{\gamma}(s'_q)$ holds. We show that both relations hold, that is,

$$\dot{\gamma}(s_q) = \dot{\gamma}(s'_q) \qquad (4.196)$$

by contradiction. Assume that one does not hold, that is, $\gamma(s_q) \prec \gamma(s'_q)$ by symmetry. Then we have

$$\begin{aligned}
f_i(s_q) &= \gamma(s_1)\gamma(s_2)\ldots\gamma(s_{q-1})\gamma(s_q) && (4.197)\\
&\overset{\text{(A)}}{=} \gamma(s'_1)\gamma(s'_2)\ldots\gamma(s'_{q-1})\gamma(s_q) && (4.198)\\
&\prec \gamma(s'_1)\gamma(s'_2)\ldots\gamma(s'_{q-1})\gamma(s'_q) && (4.199)\\
&= f_i(s'_q), && (4.200)
\end{aligned}$$

where (A) follows from the induction hypothesis. Hence, we obtain

$$s_q \overset{\text{(A)}}{\in} \mathcal{S}^{\prec}_{F,i}(f_i(s'_q)) = \{s'_1, s'_2, \ldots s'_{q-1}\} \overset{\text{(B)}}{=} \{s_1, s_2, \ldots s_{q-1}\}, \qquad (4.201)$$

where (A) follows from (4.200), and (B) follows from the induction hypothesis. This conflicts with the definition of γ-decomposition of $f_i(s'_{\rho'})$. Consequently, (4.196) holds.

Since $q = 1$ or $s_{q-1} = s'_{q-1}$ hold by the induction hypothesis and (4.196) holds, we obtain $\gamma(s_q) = \gamma(s'_q)$ by applying Lemma 4.3.3. Namely, the condition (a) holds for $r = q$.

Next, we show that the condition (b) holds for $r = q$. We have

$$f_i(s_q) = \gamma(s_1)\gamma(s_2)\ldots\gamma(s_q) \overset{\text{(A)}}{=} \gamma(s'_1)\gamma(s'_2)\ldots\gamma(s'_q) = f_i(s'_q), \tag{4.202}$$

where (A) follows from the induction hypothesis and $\gamma(s_q) = \gamma(s'_q)$ proven above. Also, if $q \neq m$, then we have $\bar{\mathcal{P}}^0_{F,i}(f_i(s_q)) \neq \emptyset$ applying Lemma 2.2.2 (i) since $f_i(s_q) \prec f_i(s_m)$. Hence, by Lemma 4.2.4 (iv), we have

$$|\mathcal{S}_{F,i}(f_i(s_q))| = 1. \tag{4.203}$$

By (4.202) and (4.203), it must hold that $s_q = s'_q$. Namely, the condition (b) holds for $r = q$. □

Proof of Lemma 4.2.7 (iii). Let $\gamma(s_1)\gamma(s_2)\ldots\gamma(s_\rho)$ (resp. $\gamma(s'_1)\gamma(s'_2)\ldots\gamma(s'_{\rho'})$) be the γ-decomposition of $f_i(s)$ (resp. $f_i(s')$).

($\Longrightarrow$) Assume $f_i(s) \preceq f_i(s')$. Then we have

$$f_i(s') = \gamma(s_1)\gamma(s_2)\ldots\gamma(s_\rho)\gamma(s'_{\rho+1})\gamma(s'_{\rho+2})\ldots\gamma(s'_{\rho'}). \tag{4.204}$$

Hence, we obtain

$$\begin{aligned} \dot{f}_i(s) &= \dot{\gamma}(s_1)\dot{\gamma}(s_2)\ldots\dot{\gamma}(s_\rho) && (4.205)\\ &\preceq \dot{\gamma}(s_1)\dot{\gamma}(s_2)\ldots\dot{\gamma}(s_\rho)\dot{\gamma}(s'_{\rho+1})\dot{\gamma}(s'_{\rho+2})\ldots\dot{\gamma}(s'_{\rho'}) && (4.206)\\ &= \dot{f}_i(s') && (4.207) \end{aligned}$$

as desired.

($\Longleftarrow$) Assume

$$\dot{f}_i(s) \preceq \dot{f}_i(s'). \tag{4.208}$$

Then we have

$$f_i(s_m) = \gamma(s_1)\gamma(s_2)\ldots\gamma(s_m) \overset{\text{(A)}}{=} \gamma(s'_1)\gamma(s'_2)\ldots\gamma(s'_m) = f_i(s'_m), \tag{4.209}$$

where $m := \{\rho, \rho'\}$ and (A) follows from Lemma 4.3.4. This implies

$$\dot{f}_i(s_m) = \dot{f}_i(s'_m) \tag{4.210}$$

by ($\Longrightarrow$) of this lemma. We consider the following two cases separately: the case $m = \rho \leq \rho'$ and the case $m = \rho' < \rho$.

- The case $m = \rho \leq \rho'$: We have

$$f_i(s) = f_i(s_m) \overset{\text{(A)}}{=} f_i(s'_m) \overset{\text{(B)}}{\preceq} f_i(s'_m)\gamma(s'_{m+1})\gamma(s'_{m+2})\ldots\gamma(s'_{\rho'}) = f_i(s') \tag{4.211}$$

as desired, where (A) follows from (4.209), and (B) follows from $m = \rho \leq \rho'$.

- The case $m = \rho' < \rho$: We show that this case is impossible. We have

$$\dot{f}_i(s_m)\dot{\gamma}(s_{m+1})\dot{\gamma}(s_{m+2})\ldots\dot{\gamma}(s_\rho) = \dot{f}_i(s) \overset{\text{(A)}}{\preceq} \dot{f}_i(s') \overset{\text{(B)}}{=} \dot{f}_i(s'_m) \overset{\text{(C)}}{=} \dot{f}_i(s_m), \tag{4.212}$$

where (A) follows from (4.208), (B) follows from $m = \rho'$, and (C) follows from (4.210). Comparing both sides, we obtain $\dot{\gamma}(s_{m+1})\dot{\gamma}(s_{m+2})\ldots\dot{\gamma}(s_\rho) = \lambda$, which leads to $\gamma(s_{m+1})\gamma(s_{m+2})\ldots\gamma(s_\rho) = \lambda$ by Lemma 4.2.7 (i). In particular, we have $\gamma(s_{m+1}) = \lambda$ by $m < \rho$. This conflicts with Lemma 4.2.6 (ii).

□

4.3.5 Proof of Lemma 4.2.8

Proof of Lemma 4.2.8. (Proof of (i) (a)): For any $\boldsymbol{x} \in \mathcal{S}^*$, we have

$$|\dot{\gamma}(s_1)| \overset{\text{(A)}}{=} |\gamma(s_1)| \overset{\text{(B)}}{\geq} 2, \tag{4.213}$$

where $\gamma(s_1)\gamma(s_2)\ldots\gamma(s_\rho)$ is the γ-decomposition of $f_i(x_1)$, and (A) follows from Lemma 4.2.7 (i), and (B) follows from $|\mathcal{P}^2_{F,i}| = 2$ and Lemma 4.2.6 (ii).

For any $c \in \mathcal{C}$, we have

$$\begin{aligned}
c \in \mathcal{P}^1_{F,i} \quad &\iff \quad {}^\exists\boldsymbol{x} \in \mathcal{S}^+; f^*_i(\boldsymbol{x}) \succeq c && (4.214)\\
&\overset{\text{(A)}}{\iff} \quad {}^\exists\boldsymbol{x} \in \mathcal{S}^+; {}^\exists c' \in \mathcal{C}; \gamma(s_1) \succeq cc' && (4.215)\\
&\overset{\text{(B)}}{\iff} \quad {}^\exists\boldsymbol{x} \in \mathcal{S}^+; \dot{\gamma}(s_1) \succeq a_{F,i}c && (4.216)\\
&\overset{\text{(C)}}{\iff} \quad {}^\exists\boldsymbol{x} \in \mathcal{S}^+; \dot{f}^*_i(\boldsymbol{x}) \succeq a_{F,i}c && (4.217)\\
&\iff \quad a_{F,i}c \in \mathcal{P}^2_{\dot{F},i}, && (4.218)
\end{aligned}$$

where $\gamma(s_1)\gamma(s_2)\ldots\gamma(s_\rho)$ is the γ-decomposition of $f_i(x_1)$, and (A) follows from (4.213), (B) follows from $|\mathcal{P}^2_{F,i}| = 2$ and the first case of (4.24), and

(C) follows from (4.213). Since $\mathcal{P}^1_{F,i} = \{0,1\}$ by $F \in \mathscr{F}_1$, we obtain $\mathcal{P}^2_{\dot{F},i} = \{a_{F,i}0, a_{F,i}1\}$ by (4.218) as desired.

(Proof of (i) (b)): Assume $|\mathcal{P}^2_{F,j}| \geq 3$. We consider the three cases of the right hand side of (4.29) separately.

- The case $|\bar{\mathcal{P}}^1_{F,i}(f_i(s))| = 0$: Clearly, we have $\mathcal{P}^2_{\dot{F},j} \subseteq \{00, 01, 10, 11\}$ as desired.
- The case $|\bar{\mathcal{P}}^1_{F,i}(f_i(s))| = 1, |\bar{\mathcal{P}}^1_{F,j}| = 1$: We have

$$\begin{aligned} 1 &\overset{\text{(A)}}{\geq} |\mathcal{S}_{F,j}(\lambda)| && (4.219)\\ &= \frac{2|\mathcal{S}_{F,j}(\lambda)|}{2} && (4.220)\\ &\overset{\text{(B)}}{=} \frac{\sum_{s \in \mathcal{S}_{F,j}(\lambda)} |\mathcal{P}^1_{F,\tau_j(s)}|}{2} && (4.221)\\ &\overset{\text{(C)}}{\geq} \frac{|\mathcal{P}^1_{F,j}| - |\bar{\mathcal{P}}^1_{F,j}|}{2} && (4.222)\\ &\overset{\text{(D)}}{=} \frac{2-1}{2} && (4.223)\\ &> 0, && (4.224) \end{aligned}$$

where (A) follows from Lemma 4.2.4 (iv) because $\bar{\mathcal{P}}^0_{F,j} \neq \emptyset$ holds by $|\bar{\mathcal{P}}^1_{F,j}| = 1$ and Corollary 2.3.1 (ii) (a), (B) follows since $|\mathcal{P}^1_{F,\tau_j(s)}| = 2$ from $F \in \mathscr{F}_1$, (C) follows from Lemma 2.2.1 (i), and (D) follows from $F \in \mathscr{F}_1$ and $|\bar{\mathcal{P}}^1_{F,j}| = 1$. Thus, we have $|\mathcal{S}_{F,j}(\lambda)| = 1$, that is, there exists $s' \in \mathcal{S}$ such that

$$\mathcal{S}_{F,j}(\lambda) = \{s'\}. \qquad (4.225)$$

Now, we have

$$|\mathcal{P}^2_{F,\tau_j(s')}| = 2 \qquad (4.226)$$

because

$$
\begin{aligned}
2 &\overset{\text{(A)}}{\leq} |\mathcal{P}^2_{F,\tau_j(s')}| && (4.227)\\
&\overset{\text{(B)}}{=} |\mathcal{P}^2_{F,j}| - |\bar{\mathcal{P}}^2_{F,j}| && (4.228)\\
&\overset{\text{(C)}}{\leq} |\mathcal{P}^2_{F,j}| - |\bar{\mathcal{P}}^1_{F,j}| && (4.229)\\
&\overset{\text{(D)}}{=} |\mathcal{P}^2_{F,j}| - 1 && (4.230)\\
&\overset{\text{(E)}}{\leq} 3 - 1 && (4.231)\\
&= 2, && (4.232)
\end{aligned}
$$

where (A) follows from Lemma 4.2.4 (i), (B) follows from Lemma 2.2.3, (C) follows from Corollary 2.3.1 (ii) (b), (D) follows from $|\bar{\mathcal{P}}^1_{F,j}| = 1$, and (E) follows from Lemma 4.2.2 and $|\bar{\mathcal{P}}^1_{F,i}(f_i(s))| = 1$.

Hence, applying the first case of (i) of this lemma, we obtain

$$\mathcal{P}^2_{\dot{F},\dot{\tau}_j(s')} = \{a_{F,\tau_j(s')}0, a_{F,\tau_j(s')}1\}. \tag{4.233}$$

Also, by (4.226) and Lemma 4.2.4 (ii) (b), we have $\bar{\mathcal{P}}^2_{F,\tau_j(s')} = \{0a, 1b\}$ for some $a, b \in \mathcal{C}$. Hence, by Lemma 2.3.2 (ii), we obtain

$$|\bar{\mathcal{P}}^1_{F,\tau_j(s')}| = |\{0, 1\}| = 2. \tag{4.234}$$

Thus, for any $\boldsymbol{x} \in \mathcal{S}^+$, we have

$$
\begin{aligned}
\dot{f}_j(x_1) &= \dot{\gamma}(s_1)\dot{\gamma}(s_2)\dots\dot{\gamma}(s_{\rho-1})\dot{\gamma}(s_\rho) && (4.235)\\
&\succeq \dot{\gamma}(s_1)\dot{\gamma}(s_2) && (4.236)\\
&\overset{\text{(A)}}{=} \dot{\gamma}(s')\dot{\gamma}(s_2) && (4.237)\\
&\overset{\text{(B)}}{\succeq} \dot{\gamma}(s')\bar{a}_{F,\tau_j(s')}1 && (4.238)\\
&\overset{\text{(C)}}{=} \bar{a}_{F,\tau_j(s')}1, && (4.239)
\end{aligned}
$$

where $\gamma(s_1)\gamma(s_2)\dots\gamma(s_{\rho-1})\gamma(s_\rho)$ is the γ-decomposition of $f_j(x_1)$, and (A) follows from (4.225) and Lemma 4.2.6 (i), (B) is obtained by applying the fifth case of (4.24) by $|\bar{\mathcal{P}}^1_{F,j}(f_j(s'))| = |\bar{\mathcal{P}}^1_{F,j}| = 1$, (4.226) and (4.234), and (C) follows from (4.225) and Lemma 4.2.7 (i). This shows

$$\bar{\mathcal{P}}^2_{\dot{F},j} \subseteq \{\bar{a}_{F,\tau_j(s')}1\}. \tag{4.240}$$

Finally, we obtain

$$\mathcal{P}^2_{\dot{F},j} \overset{\text{(A)}}{=} \bar{\mathcal{P}}^2_{\dot{F},j} \cup \Big(\bigcup_{s'' \in \mathcal{S}_{F,j}(\lambda)} \mathcal{P}^2_{\dot{F},\dot{\tau}_j(s'')} \Big) \tag{4.241}$$

$$\overset{\text{(B)}}{=} \bar{\mathcal{P}}^2_{\dot{F},j} \cup \mathcal{P}^2_{\dot{F},\dot{\tau}_j(s')} \tag{4.242}$$

$$\overset{\text{(C)}}{\subseteq} \{a_{F,\tau_j(s')}0, a_{F,\tau_j(s')}1, \bar{a}_{F,\tau_j(s')}1\} \tag{4.243}$$

$$\overset{\text{(D)}}{=} \{a_{F,j}0, a_{F,j}1, \bar{a}_{F,j}1\} \tag{4.244}$$

as desired, where (A) follows from Lemma 2.2.1 (i), (B) follows from (4.225), (C) follows from (4.233) and (4.240), and (D) follows since $a_{F,\tau_j(s')} = a_{F,j}$ by (4.225) and the first case of (4.25).

- The case $|\bar{\mathcal{P}}^1_{F,i}(f_i(s))| = 1, |\bar{\mathcal{P}}^1_{F,j}| = 2$: We show $\boldsymbol{c} \in \mathcal{P}^2_{F,j}$ for an arbitrarily fixed $\boldsymbol{c} = c_1c_2 \in \mathcal{P}^2_{\dot{F},j}$.

  We have

$$|\mathcal{S}_{F,j}(\lambda)| = \frac{2|\mathcal{S}_{F,j}(\lambda)|}{2} \tag{4.245}$$

$$\overset{\text{(A)}}{\leq} \frac{\sum_{s' \in \mathcal{S}_{F,j}(\lambda)} |\mathcal{P}^2_{F,\tau_j(s')}|}{2} \tag{4.246}$$

$$\overset{\text{(B)}}{=} \frac{|\mathcal{P}^2_{F,j}| - |\bar{\mathcal{P}}^2_{F,j}|}{2} \tag{4.247}$$

$$\overset{\text{(C)}}{\leq} \frac{|\mathcal{P}^2_{F,j}| - |\bar{\mathcal{P}}^1_{F,j}|}{2} \tag{4.248}$$

$$\overset{\text{(D)}}{\leq} \frac{3 - |\bar{\mathcal{P}}^1_{F,j}|}{2} \tag{4.249}$$

$$\overset{\text{(E)}}{=} \frac{3-2}{2} \tag{4.250}$$

$$< 1, \tag{4.251}$$

where (A) follows since $|\mathcal{P}^2_{F,\tau_j(s')}| \geq 2$ for any $s' \in \mathcal{S}_{F,j}(\lambda)$ by Lemma 4.2.4 (i), (B) follows from Lemma 2.2.3, (C) follows from Corollary 2.3.1 (ii) (b), (D) follows from Lemma 4.2.2 and $|\bar{\mathcal{P}}^1_{F,i}(f_i(s))| = 1$, and (E) follows from $|\bar{\mathcal{P}}^1_{F,j}| = 2$. This shows

$$\mathcal{S}_{F,j}(\lambda) = \emptyset. \tag{4.252}$$

By $\boldsymbol{c} \in \mathcal{P}^2_{\dot{F},j}$, there exists $\boldsymbol{x} \in \mathcal{S}^+$ such that

$$\dot{f}^*_j(\boldsymbol{x}) \succeq \boldsymbol{c}. \tag{4.253}$$

Then we have

$$\dot{f}_j(x_1) = \dot{\gamma}(s_1)\dot{\gamma}(s_2)\ldots\dot{\gamma}(s_{\rho-1})\dot{\gamma}(s_\rho) \succeq \dot{\gamma}(s_1) \overset{\text{(A)}}{=} \gamma(s_1), \tag{4.254}$$

where $\gamma(s_1)\gamma(s_2)\ldots\gamma(s_{\rho-1})\gamma(s_\rho)$ is the γ-decomposition $f_j(x_1)$ and (A) follows from $|\mathcal{P}^2_{\dot{F},j}| \geq 3$ and the second case of (4.24).

By (4.252) and Lemma 4.2.6 (i), it holds that $|\gamma(s_1)| \geq 1$. We consider the following two cases separately: the case $|\gamma(s_1)| = 1$ and the case $|\gamma(s_1)| \geq 2$.

- The case $|\gamma(s_1)| = 1$: By (4.253) and (4.254), we have

$$f_j(s_1) = \gamma(s_1) = c_1. \tag{4.255}$$

We obtain

$$\begin{aligned}
\mathcal{P}^2_{F,j} &\overset{\text{(A)}}{\supseteq} \bar{\mathcal{P}}^2_{F,j} &(4.256)\\
&\overset{\text{(B)}}{\supseteq} c_1\mathcal{P}^1_{F,j}(c_1) &(4.257)\\
&\overset{\text{(C)}}{=} c_1\mathcal{P}^1_{F,j}(f_j(s_1)) &(4.258)\\
&\overset{\text{(D)}}{\supseteq} c_1\mathcal{P}^1_{F,\tau_j(s_1)} &(4.259)\\
&\overset{\text{(E)}}{=} c_1\{0,1\} &(4.260)\\
&\ni c_1c_2 &(4.261)\\
&= \boldsymbol{c} &(4.262)
\end{aligned}$$

as desired, where (A) follows from Lemma 2.2.1 (i), (B) follows from Lemma 2.2.1 (ii), (C) follows from (4.255), (D) follows from Lemma 2.2.1 (i), and (E) follows from $F \in \mathscr{F}_1$.

- The case $|\gamma(s_1)| \geq 2$: By (4.253) and (4.254), we have $f^*_j(\boldsymbol{x}) \succeq \gamma(s_1) \succeq \boldsymbol{c}$, which leads to $\boldsymbol{c} \in \mathcal{P}^2_{F,j}$.

(Proof of (ii)): We consider the following two cases separately: the case $|\bar{\mathcal{P}}^1_{F,i}(f_i(s))| = 0$ and the $|\bar{\mathcal{P}}^1_{F,i}(f_i(s))| \geq 1$.

- The case $|\bar{\mathcal{P}}^1_{F,i}(f_i(s))| = 0$: We have

$$
\begin{aligned}
|\bar{\mathcal{P}}^1_{F,i}(f_i(s))| = 0 \quad &\overset{\text{(A)}}{\iff} \quad \bar{\mathcal{P}}^0_{F,i}(f_i(s)) = \emptyset &(4.263)\\
&\overset{\text{(B)}}{\iff} \quad {}^\forall s' \in \mathcal{S}; f_i(s) \not\succ f_i(s') &(4.264)\\
&\overset{\text{(C)}}{\iff} \quad {}^\forall s' \in \mathcal{S}; \dot{f}_i(s) \not\succ \dot{f}_i(s') &(4.265)\\
&\overset{\text{(D)}}{\iff} \quad \bar{\mathcal{P}}^0_{\dot{F},i}(\dot{f}_i(s)) = \emptyset &(4.266)\\
&\overset{\text{(E)}}{\iff} \quad \bar{\mathcal{P}}^2_{\dot{F},i}(\dot{f}_i(s)) = \emptyset &(4.267)
\end{aligned}
$$

as desired, where (A) follows from Corollary 2.3.1 (ii) (a), (B) follows from Lemma 2.2.2 (i), (C) follows from Lemma 4.2.7 (iii), (D) follows from Lemma 2.2.2 (i), and (E) follows from Corollary 2.3.1 (ii) (a).

- The case $|\bar{\mathcal{P}}^1_{F,i}(f_i(s))| \geq 1$: Choose $\boldsymbol{x} \in \mathcal{S}^+$ such that $\dot{f}^*_i(\boldsymbol{x}) \succeq \dot{f}_i(s)$, and $\dot{f}_i(x_1) \succ \dot{f}_i(s)$ arbitrary and let $\gamma(s_1)\gamma(s_2)\ldots\gamma(s_{\rho'})$ be the γ-decomposition of $f_i(x_1)$. Then by $\dot{f}_i(x_1) \succ \dot{f}_i(s)$, there exists an integer ρ such that $\rho < \rho'$ and $f_i(s) = \gamma(s_1)\gamma(s_2)\ldots\gamma(s_\rho)$. We have

$$
\begin{aligned}
\dot{f}^*_i(\boldsymbol{x}) \quad &\succeq \quad \dot{f}_i(x_1) &(4.268)\\
&= \quad \dot{\gamma}(s_1)\dot{\gamma}(s_2)\ldots\dot{\gamma}(s_{\rho'}) &(4.269)\\
&= \quad \dot{f}_i(s)\dot{\gamma}(s_{\rho+1})\ldots\dot{\gamma}(s_{\rho'}) &(4.270)\\
&\succeq \quad \dot{f}_i(s)\dot{\gamma}(s_{\rho+1}) &(4.271)\\
&\overset{\text{(A)}}{\succeq} \quad \dot{f}_i(s)\dot{g}_1\dot{g}_2, &(4.272)
\end{aligned}
$$

where $\dot{\gamma}(s_{\rho+1}) = \dot{g}_1\dot{g}_2\ldots\dot{g}_l$, and (A) follows since $|\dot{\gamma}(s_{\rho+1})| = |\gamma(s_{\rho+1})| \geq 2$ by Lemma 4.2.6 (ii) and Lemma 4.2.7 (i). Therefore, the set $\bar{\mathcal{P}}^2_{\dot{F},i}(\dot{f}_i(s))$ is included in the set of all possible sequences as $\dot{g}_1\dot{g}_2 \in \mathcal{C}^2$. We consider what sequences are possible as $\dot{g}_1\dot{g}_2 \in \mathcal{C}^2$ for the following three cases: the case $|\mathcal{P}^2_{F,j}| = 2$, the case $|\mathcal{P}^2_{F,j}| \geq 3, |\bar{\mathcal{P}}^1_{F,j}| = 1$, and the case $|\mathcal{P}^2_{F,j}| \geq 3, |\bar{\mathcal{P}}^1_{F,j}| = 2$.

  - The case $|\mathcal{P}^2_{F,j}| = 2$:
    - The case $|\bar{\mathcal{P}}^1_{F,i}(f_i(s))| = 2$: We have $\dot{g}_1\dot{g}_2 \subseteq \{\bar{a}_{F,j}0, \bar{a}_{F,j}1\}$ applying the third case of (4.24).
    - The case $|\bar{\mathcal{P}}^1_{F,i}(f_i(s))| = 1$: By $|\mathcal{P}^2_{F,j}| = 2$ and Lemma 4.2.4 (ii) (b), we have $|\mathcal{P}^2_{F,j}| = \{0a, 1b\}$ for some $a, b \in \mathcal{C}$. Thus, we

have $|\bar{\mathcal{P}}^1_{F,j}| = |\{0,1\}| = 2$ applying Lemma 2.3.2 (ii). Hence, we obtain $\dot{g}_1\dot{g}_2 = \bar{a}_{F,j}1$ applying the fifth case of (4.24).

These show $\bar{\mathcal{P}}^2_{\dot{F},i}(\dot{f}_i(s)) \subseteq \{\bar{a}_{F,j}0, \bar{a}_{F,j}1\}$ as desired.

– The case $|\mathcal{P}^2_{F,j}| \geq 3$: Then we have $|\bar{\mathcal{P}}^1_{F,i}(f_i(s))| \leq 1$ by Lemma 4.2.2. Combining this with $|\bar{\mathcal{P}}^1_{F,i}(f_i(s))| \geq 1$, we obtain

$$|\bar{\mathcal{P}}^1_{F,i}(f_i(s))| = 1. \tag{4.273}$$

  * The case $|\bar{\mathcal{P}}^1_{F,j}| = 1$: We obtain $\dot{g}_1\dot{g}_2 = \bar{a}_{F,j}0$ applying the fourth case of (4.24) by (4.273) and $|\bar{\mathcal{P}}^1_{F,j}| = 1$. This shows $\bar{\mathcal{P}}^2_{\dot{F},i}(\dot{f}_i(s)) \subseteq \{\bar{a}_{F,j}0\}$ as desired.
  * The case $|\bar{\mathcal{P}}^1_{F,j}| = 2$: We obtain $\dot{g}_1\dot{g}_2 = g_1g_2$ by the sixth case of (4.24) by (4.273), $|\bar{\mathcal{P}}^1_{F,j}| = 2$, and $|\mathcal{P}^2_{F,j}| \geq 3$. This shows $\bar{\mathcal{P}}^2_{\dot{F},i}(\dot{f}_i(s)) \subseteq \bar{\mathcal{P}}^2_{F,i}(f_i(s))$ as desired because $g_1g_2 \in \bar{\mathcal{P}}^2_{F,i}(f_i(s))$ by Lemma 4.2.6 (iii).

□

4.3.6 Proof of Lemma 4.2.9

Proof of Lemma 4.2.9. (Proof of (i)): Assume $|\mathcal{P}^2_{F,i}| = 2$. Then we have $\mathcal{P}^2_{\dot{F},i} = \{a_{F,i}0, a_{F,i}1\}$ by Lemma 4.2.8 (i) (a). Hence, we have $\mathcal{P}^1_{\dot{F},i} = \{a_{F,i}\}$ by Lemma 2.3.2 (i). Therefore, by (3.123), we obtain $d_{\dot{F},i} = a_{F,i}$ as desired.

(Proof of (ii)): Assume $s \neq s'$ and $\dot{f}_i(s) = \dot{f}_i(s')$. Then since $f_i(s) = f_i(s')$ by Lemma 4.2.7 (iii), we have

$$|\mathcal{P}^2_{F,\tau_i(s)}| = |\mathcal{P}^2_{F,\tau_i(s')}| = 2 \tag{4.274}$$

applying Lemma 4.2.4 (iii). Hence, by (i) of this lemma, we obtain

$$d_{\dot{F},\tau_i(s)} = a_{F,\tau_i(s)}, \quad d_{\dot{F},\tau_i(s')} = a_{F,\tau_i(s')}. \tag{4.275}$$

Also, by (4.274) and Lemma 4.2.4 (ii) (a), we have $\mathcal{S}_{F,\tau_i(s)}(\lambda) = \mathcal{S}_{F,\tau_i(s')}(\lambda) = \emptyset$, in particular,

$$|\mathcal{S}_{F,\tau_i(s)}(\lambda)| \neq 1, \quad |\mathcal{S}_{F,\tau_i(s')}(\lambda)| \neq 1. \tag{4.276}$$

Now we show $\mathcal{P}^2_{F,\tau_i(s)} \ni 0a_{F,\tau_i(s)}$ considering the following two cases: the case $\mathcal{P}^2_{F,\tau_i(s)} \ni 00$ and the case $\mathcal{P}^2_{F,\tau_i(s)} \not\ni 00$.

- The case $\mathcal{P}^2_{F,\tau_i(s)} \ni 00$: By (4.276) and the second case of (4.25), we have $a_{F,\tau_i(s)} = 0$ and thus $\mathcal{P}^2_{F,\tau_i(s)} \ni 00 = 0a_{F,\tau_i(s)}$.

- The case $\mathcal{P}^2_{F,\tau_i(s)} \not\ni 00$: By Lemma 4.2.4 (ii) (b), there exists $b \in \mathcal{C}$ such that
$$\mathcal{P}^2_{F,\tau_i(s)} \ni 0b \overset{\text{(A)}}{=} 01 \overset{\text{(B)}}{=} 0a_{F,\tau_i(s)}, \tag{4.277}$$
where (A) follows from $\mathcal{P}^2_{F,\tau_i(s)} \not\ni 00$, and (B) follows from (4.276), $\mathcal{P}^2_{F,\tau_i(s)} \not\ni 00$, and the third case of (4.25).

Therefore, we conclude that $\mathcal{P}^2_{F,\tau_i(s)} \ni 0a_{F,\tau_i(s)}$. By the same argument, we also have $\mathcal{P}^2_{F,\tau_i(s')} \ni 0a_{F,\tau_i(s')}$. Consequently, we have

$$\{0a_{F,\tau_i(s)}\} \cap \{0a_{F,\tau_i(s')}\} \subseteq \mathcal{P}^2_{F,\tau_i(s)} \cap \mathcal{P}^2_{F,\tau_i(s')} \overset{\text{(A)}}{=} \emptyset, \tag{4.278}$$

where (A) follows from $F \in \mathscr{F}_{2\text{-dec}}$. This shows

$$a_{F,\tau_i(s)} \neq a_{F,\tau_i(s')}. \tag{4.279}$$

Combining (4.275) and (4.279), we obtain the desired result. □

4.3.7 Proof of Lemma 4.2.11

To prove Lemma 4.2.11, we prove Lemmas 4.3.5 and 4.3.6 as follows.

Lemma 4.3.5. *For any $F \in \mathscr{F}_1$ and $i \in [F]$, the mapping $\widehat{\dot{f}_i}$ is injective.*

Proof of Lemma 4.3.5. Choose $s, s' \in \mathcal{S}$ such that $\widehat{\dot{f}_i}(s) = \widehat{\dot{f}_i}(s')$ arbitrarily. We show $s = s'$.

We have

$$\dot{f}_i(s) d_{\dot{F},\dot{\tau}_i(s)} \overset{\text{(A)}}{=} d_{\dot{F},i}\widehat{\dot{f}_i}(s) \overset{\text{(B)}}{=} d_{\dot{F},i}\widehat{\dot{f}_i}(s') \overset{\text{(C)}}{=} \dot{f}_i(s') d_{\dot{F},\dot{\tau}_i(s')}, \tag{4.280}$$

where (A) follows from Lemma 3.4.1 (i), (B) follows directly from $\widehat{\dot{f}_i}(s) = \widehat{\dot{f}_i}(s')$, and (C) follows from Lemma 3.4.1 (i).

Also, we have
$$|d_{\dot{F},\dot{\tau}_i(s)}| = |d_{\dot{F},\dot{\tau}_i(s')}| \tag{4.281}$$

because if we assume the contrary, that is, $|d_{\dot{F},\dot{\tau}_i(s)}| = 1$ and $|d_{\dot{F},\dot{\tau}_i(s')}| = 0$ by symmetry, then by (4.280), we have $\dot{f}_i(s)d_{\dot{F},\dot{\tau}_i(s)} = \dot{f}_i(s')$, which conflicts with Lemma 4.2.4 (v).

By (4.280) and (4.281), we obtain $\dot{f}_i(s) = \dot{f}_i(s')$ and $d_{\dot{F},\dot{\tau}_i(s)} = d_{\dot{F},\dot{\tau}_i(s')}$. Hence, we obtain $s = s'$ as desired applying the contraposition of Lemma 4.2.9 (ii). □

Lemma 4.3.6. *For any $F \in \mathscr{F}_1, i \in [F]$, and $s \in \mathcal{S}$, if $\bar{\mathcal{P}}^0_{F,i}(f_i(s)) = \emptyset$ or $\tau_i(s) \in \mathcal{M}_F$, then $\bar{\mathcal{P}}^0_{\hat{\dot{F}},i}(\widehat{\dot{f}}_i(s)) = \emptyset$.*

Proof of Lemma 4.3.6. We assume that $\bar{\mathcal{P}}^0_{F,i}(f_i(s)) = \emptyset$ or $\tau_i(s) \in \mathcal{M}_F$ holds and prove by contradiction assuming $\bar{\mathcal{P}}^0_{\hat{\dot{F}},i}(\widehat{\dot{f}}_i(s)) \neq \emptyset$. Then by Lemma 2.2.2 (i), there exist $s' \in \mathcal{S} \setminus \{s\}$ and $c \in \mathcal{C}$ such that

$$\widehat{\dot{f}}_i(s)c \preceq \widehat{\dot{f}}_i(s'). \tag{4.282}$$

Thus, we have

$$\dot{f}_i(s)d_{\dot{F},\dot{\tau}_i(s)}c \overset{\text{(A)}}{=} d_{\dot{F},i}\widehat{\dot{f}}_i(s)c \overset{\text{(B)}}{\preceq} d_{\dot{F},i}\widehat{\dot{f}}_i(s') \overset{\text{(C)}}{=} \dot{f}_i(s')d_{\dot{F},\dot{\tau}_i(s')}, \tag{4.283}$$

where (A) follows from Lemma 3.4.1 (i), (B) follows from (4.282), and (C) follows from Lemma 3.4.1 (i).

We consider the following two cases separately: the case $\bar{\mathcal{P}}^0_{F,i}(f_i(s)) = \emptyset$ and the case $\tau_i(s) \in \mathcal{M}_F$.

- The case $\bar{\mathcal{P}}^0_{F,i}(f_i(s)) = \emptyset$: We have

  $$|\bar{\mathcal{P}}^0_{\dot{F},i}(\dot{f}_i(s))| \overset{\text{(A)}}{\leq} |\bar{\mathcal{P}}^2_{\dot{F},i}(\dot{f}_i(s))| \overset{\text{(B)}}{=} 0, \tag{4.284}$$

  where (A) follows from Corollary 2.3.1 (ii) (b), and (B) follows from the first case of (4.30) because $\bar{\mathcal{P}}^1_{F,i}(f_i(s)) = \emptyset$ holds by $\bar{\mathcal{P}}^0_{F,i}(f_i(s)) = \emptyset$ and Corollary 2.3.1 (ii) (a).

  Also, we have

  $$|\dot{f}_i(s)| \overset{\text{(A)}}{\leq} |\dot{f}_i(s')| + |d_{\dot{F},\dot{\tau}_i(s')}| - |d_{\dot{F},\dot{\tau}_i(s)}| - |c| \overset{\text{(B)}}{\leq} |\dot{f}_i(s')|, \tag{4.285}$$

where (A) follows from (4.283), and (B) follows from $|d_{\dot{F},\dot{\tau}_i(s')}| \leq 1$, $|d_{\dot{F},\dot{\tau}_i(s)}| \geq 0$, and $|c| = 1$.

In fact, the equalities hold in (4.285), that is, we have

$$|\dot{f}_i(s)| = |\dot{f}_i(s')| \tag{4.286}$$

because if we assume $|\dot{f}_i(s)| < |\dot{f}_i(s')|$, then we have $\dot{f}_i(s) \prec \dot{f}_i(s')$ by (4.283), which conflicts with (4.284) and Lemma 2.2.2 (i).

By (4.283) and (4.286), we obtain

$$\dot{f}_i(s) = \dot{f}_i(s'). \tag{4.287}$$

Hence, applying Lemma 4.2.9 (ii), we have $d_{\dot{F},\dot{\tau}_i(s)} = a_{F,\tau_i(s)}$ and $d_{\dot{F},\dot{\tau}_i(s')} = a_{F,\tau_i(s')}$. In particular,

$$|d_{\dot{F},\dot{\tau}_i(s)}| = |d_{\dot{F},\dot{\tau}_i(s')}| = 1. \tag{4.288}$$

Thus, we obtain

$$|\dot{f}_i(s)|+2 \overset{\text{(A)}}{=} |\dot{f}_i(s) d_{\dot{F},\dot{\tau}_i(s)} c| \overset{\text{(B)}}{\leq} |\dot{f}_i(s') d_{\dot{F},\dot{\tau}_i(s')}| \overset{\text{(C)}}{=} |\dot{f}_i(s')|+1 \overset{\text{(D)}}{=} |\dot{f}_i(s)|+1, \tag{4.289}$$

where (A) follows from (4.288), (B) follows from (4.283), (C) follows from (4.288), and (D) follows from (4.287). This is a contradiction.

- The case $\tau_i(s) \in \mathcal{M}_F$: By Lemma 4.2.9 (i), we have

$$d_{\dot{F},\dot{\tau}_i(s)} = a_{F,\tau_i(s)}. \tag{4.290}$$

Substituting (4.290) for (4.283), we obtain

$$\dot{f}_i(s) a_{F,\tau_i(s)} c \preceq \dot{f}_i(s') d_{\dot{F},\dot{\tau}_i(s')}. \tag{4.291}$$

Also, we have

$$|\dot{f}_i(s)| + 1 = |\dot{f}_i(s)| + |a_{F,\tau_i(s)}| \overset{\text{(A)}}{\leq} |\dot{f}_i(s')| + |d_{\dot{F},\dot{\tau}_i(s')}| - |c| \overset{\text{(B)}}{\leq} |\dot{f}_i(s')|, \tag{4.292}$$

where (A) follows from (4.291), and (B) follows from $|d_{\dot{F},\dot{\tau}_i(s')}| \leq 1$ and $|c| = 1$.

By (4.291) and (4.292), we have $\dot{f}_i(s)a_{F,\tau_i(s)} \preceq \dot{f}_i(s')$, which leads to $\bar{\mathcal{P}}^1_{\dot{F},i}(\dot{f}_i(s)) \ni a_{F,\tau_i(s)}$. Hence, applying Lemma 2.3.2 (ii), we have

$$\bar{\mathcal{P}}^2_{\dot{F},i}(\dot{f}_i(s)) \ni a_{F,\tau_i(s)}c' \tag{4.293}$$

for some $c' \in \mathcal{C}$. On the other hand, by $\tau_i(s) \in \mathcal{M}_F$ and Lemma 4.2.8 (i) (a), we have

$$\mathcal{P}^2_{\dot{F},\dot{\tau}_i(s)} = \{a_{F,\tau_i(s)}0, a_{F,\tau_i(s)}1\}. \tag{4.294}$$

By (4.293) and (4.294), we obtain $\mathcal{P}^2_{\dot{F},\dot{\tau}_i(s)} \cap \bar{\mathcal{P}}^2_{\dot{F},i}(\dot{f}_i(s)) \neq \emptyset$. Hence, we have $\dot{F} \notin \mathscr{F}_{2\text{-dec}}$, which conflicts with the proof of Lemma 4.2.10.

□

Proof of Lemma 4.2.11. Applying Lemma 4.2.10 in a repetitive manner, we have

$$F^{(0)}, F^{(1)}, \ldots, F^{(t)}, F^{(t+1)}, \ldots, F^{(t')} \in \mathscr{F}_1 \tag{4.295}$$

and

$$L(F) = L(F^{(0)}) = L(F^{(1)}) = \cdots = L(F^{(t)}) = L(F^{(t+1)}) = \cdots = L(F^{(t')}). \tag{4.296}$$

We prove Lemma 4.2.11 by contradiction assuming that there exists $p \in \mathcal{M}_{F^{(t)}} \cap \mathcal{M}_{F^{(t')}}$. By $\mathcal{R}_F = |F|$, there exist $i \in [F]$ and $s \in \mathcal{S}$ such that $\tau_i(s) = p$. By (3.122) and (4.23), we have $\tau_i^{(t)}(s) = \tau_i^{(t')}(s) = p$ and

$$\begin{aligned}
&\tau_i^{(t)}(s) = p \in \mathcal{M}_{F^{(t)}} \\
&\overset{\text{(A)}}{\Longrightarrow} \quad \bar{\mathcal{P}}^0_{F^{(t+1)},i}(f_i^{(t+1)}(s)) = \emptyset && (4.297)\\
&\overset{\text{(A)}}{\Longrightarrow} \quad \bar{\mathcal{P}}^0_{F^{(t+2)},i}(f_i^{(t+1)}(s)) = \emptyset && (4.298)\\
&\overset{\text{(A)}}{\Longrightarrow} \quad \cdots && (4.299)\\
&\overset{\text{(A)}}{\Longrightarrow} \quad \bar{\mathcal{P}}^0_{F^{(t')},i}(f_i^{(t')}(s)) = \emptyset, && (4.300)
\end{aligned}$$

where (A)s follow from (4.295) and Lemma 4.3.6. Applying Lemma 4.3.5 to $F^{(t'-1)}$, we see that $f_i^{(t')}(s)$ is injective, in particular,

$$|\mathcal{S}_{F^{(t')},i}(f_i^{(t')}(s))| = 1. \tag{4.301}$$

By (4.300) and (4.301), we obtain $|\mathcal{P}^2_{F^{(t')},p}| = |\mathcal{P}^2_{F^{(t')},\tau_i^{(t')}(s)}| = 4$ applying Lemma 4.2.5, which conflicts with $p \in \mathcal{M}_{F^{(t')}}$. □

4.3.8 Proof of Lemma 4.2.15 (iii)

We can prove Lemma 4.2.15 (iii) in a similar way to prove Lemma 4.2.7 (iii) by using the following Lemma 4.3.7 instead of Lemma 4.3.3.

Lemma 4.3.7. *Let $F \in \mathscr{F}_2$, $i \in [F]$, and $s, s' \in \mathcal{S}$, and let $\gamma(s_1)\gamma(s_2)\ldots\gamma(s_\rho)$ (resp. $\gamma(s'_1)\gamma(s'_2)\ldots\gamma(s'_{\rho'})$) be the γ-decomposition of $f_i(s)$ (resp. $f_i(s')$). For any $r = 1, 2, \ldots, m := \min\{\rho, \rho'\}$, if one of the following conditions (a) and (b) holds, then $\gamma(s_r) = \gamma(s'_r) \iff \ddot{\gamma}(s_r) = \ddot{\gamma}(s'_r)$:*

(a) $r = 1$.

(b) $r \geq 2$ *and* $s_{r-1} = s'_{r-1}$.

Proof of Lemma 4.3.3. Assume that (a) or (b) holds.

($\Longrightarrow$) Directly from (4.55).

($\Longleftarrow$) We prove the contraposition. Namely, we prove $\ddot{\gamma}(s_r) \neq \ddot{\gamma}(s'_r)$ assuming $\gamma(s_r) \neq \gamma(s'_r)$. Put $\gamma(s_r) = g_1 g_2 \ldots g_l$ and $\gamma(s'_r) = g'_1 g'_2 \ldots g'_{l'}$. We consider the following two cases separately: the case $|\gamma(s_r)| \neq |\gamma(s'_r)|$ and the case $|\gamma(s_r)| = |\gamma(s'_r)|$.

- The case $|\gamma(s_r)| \neq |\gamma(s'_r)|$: We have

$$|\ddot{\gamma}(s_r)| \overset{\text{(A)}}{=} |\gamma(s_r)| \overset{\text{(B)}}{\neq} |\gamma(s'_r)| \overset{\text{(C)}}{=} |\ddot{\gamma}(s'_r)|, \tag{4.302}$$

  where (A) follows from Lemma 4.2.15 (i), (B) follows from the assumption, and (C) follows from Lemma 4.2.15 (i). This shows $\ddot{\gamma}(s_r) \neq \ddot{\gamma}(s'_r)$.

- The case $|\gamma(s_r)| = |\gamma(s'_r)|$: If $|\gamma(s_r)| = |\gamma(s'_r)| \geq 3$ and $g_3 g_4 \ldots g_l \neq g'_3 g'_4 \ldots g'_{l'}$, then we obtain $\ddot{\gamma}(s_r) \neq \ddot{\gamma}(s'_r)$ directly from (4.55). Thus, we assume

$$g_j \neq g'_j \text{ for some } 1 \leq j \leq \min\{2, |\gamma(s_r)|\}. \tag{4.303}$$

  Now we show that the condition (a) necessarily holds by contradiction assuming that the condition (a) does not hold and the condition (b) holds. Then we have $|\gamma(s_r)| = |\gamma(s'_r)| \geq 2$ by Lemma 4.2.6 (ii) and we have $g_1 g_2 \in \bar{\mathcal{P}}^2_{F,i}(f_i(s_{r-1}))$ and $g'_1 g'_2 \in \bar{\mathcal{P}}^2_{F,i}(f_i(s'_{r-1}))$ by Lemma 4.2.6 (iii). Since $s_{r-1} = s'_{r-1}$ by the condition (b), we have

$$\{g_1 g_2, g'_1 g'_2\} \subseteq \bar{\mathcal{P}}^2_{F,i}(f_i(s_{r-1})). \tag{4.304}$$

Therefore, we have

$$\begin{aligned}
|\{g_1g_2, g'_1g'_2\}| &\overset{(A)}{\leq} |\bar{\mathcal{P}}^2_{F,i}(f_i(s_{r-1}))| && (4.305)\\
&\overset{(B)}{\leq} |\mathcal{P}^2_{F,i}(f_i(s_{r-1}))| - |\mathcal{P}^2_{F,\tau_i(s_{r-1})}| && (4.306)\\
&\overset{(C)}{\leq} 4-3 && (4.307)\\
&= 1, && (4.308)
\end{aligned}$$

where (A) follows from (4.304), (B) follows from Lemma 2.2.3, and (C) follows from $F \in \mathscr{F}_2$. This leads to $g_1g_2 = g'_1g'_2$, which conflicts with (4.303). Therefore, the condition (a), that is, $r = 1$ holds.

We consider the following two cases separately: the case $|\mathcal{P}^2_{F,i}| = 4$ and the case $|\mathcal{P}^2_{F,i}| = 3$.

- The case $|\mathcal{P}^2_{F,i}| = 4$: We obtain

$$\ddot{\gamma}(s_1) \overset{(A)}{=} \gamma(s_1) \overset{(B)}{\neq} \gamma(s'_1) \overset{(C)}{=} \ddot{\gamma}(s'_1) \qquad (4.309)$$

  as desired, where (A) follows from $|\mathcal{P}^2_{F,i}| = 4$ and the first case of (4.55), (B) follows from (4.303), and (C) follows from $|\mathcal{P}^2_{F,i}| = 4$ and the first case of (4.55).

- The case $|\mathcal{P}^2_{F,i}| = 3$: We first prove

$$|\gamma(s_1)| = |\gamma(s'_1)| \geq 2 \qquad (4.310)$$

  by assuming the contrary $|\gamma(s_1)| = |\gamma(s'_1)| = 1$. Then by (4.303), we may assume $\gamma(s_1) = 0$ and $\gamma(s'_1) = 1$ without loss of generality. Hence, we have

$$\begin{aligned}
\mathcal{P}^2_{F,i} &\overset{(A)}{\supseteq} \bar{\mathcal{P}}^2_{F,i} && (4.311)\\
&\overset{(B)}{=} 0\mathcal{P}^1_{F,i}(0) \cup 1\mathcal{P}^1_{F,i}(1) && (4.312)\\
&\overset{(C)}{\supseteq} 0\mathcal{P}^1_{F,\tau_i(s_1)} \cup 1\mathcal{P}^1_{F,\tau_i(s'_1)} && (4.313)\\
&\overset{(D)}{=} 0\{0,1\} \cup 1\{0,1\} && (4.314)\\
&= \{00, 01, 10, 11\}, && (4.315)
\end{aligned}$$

where (A) follows from Lemma 2.2.1 (i), (B) follows from Lemma 2.2.1 (ii), (C) follows from Lemma 2.2.1 (i), and (D) follows from $F \in \mathscr{F}_2 \subseteq \mathscr{F}_1$. This conflicts with $|\mathcal{P}^2_{F,i}| = 3$. Therefore, (4.310) holds.

By $|\mathcal{P}^2_{F,i}| = 3$, we have $\mathcal{P}^2_{F,i} = \{h_1h_2, \bar{h}_10, \bar{h}_11\}$ for some $h_1h_2 \in \mathcal{C}^2$. By (4.310), we have $g_1g_2 \in \mathcal{P}^2_{F,i} = \{h_1h_2, \bar{h}_10, \bar{h}_11\}$.

* If $g_1g_2 = h_1h_2$, then $\ddot{\gamma}(s_1) = 01$ by the third case of (4.55).
* If $g_1g_2 = \bar{h}_10$, then $\ddot{\gamma}(s_1) = 10$ by the fourth case of (4.55).
* If $g_1g_2 = \bar{h}_11$, then $\ddot{\gamma}(s_1) = 11$ by the fourth case of (4.55).

By the same argument, we have $\ddot{\gamma}(s'_1) = 01$ (resp. $10, 11$) if $g'_1g'_2 = h_1h_2$ (resp. $\bar{h}_10, \bar{h}_11$). In particular, $\ddot{\gamma}(s_1) = \ddot{\gamma}(s'_1)$ holds if and only if $g_1g_2 = g'_1g'_2$. Therefore, $\ddot{\gamma}(s_1) \neq \ddot{\gamma}(s'_1)$ is implied by (4.303) as desired.

□

4.3.9 Proof of Lemma 4.2.16

Proof of Lemma 4.2.16. (Proof of (i)): We consider the following two cases separately: (I) the case $|\mathcal{P}^2_{F,i}| = 3$; (II) the case $|\mathcal{P}^2_{F,i}| = 4$.

(I) The case $|\mathcal{P}^2_{F,i}| = 3$: Choose $\boldsymbol{x} \in \mathcal{S}^*$ arbitrarily, and let $\gamma(s_1)\gamma(s_2)\ldots\gamma(s_\rho)$ be the γ-decomposition of $f_i(x_1)$. By $|\mathcal{P}^2_{F,i}| = 3$, applying the second, third, and fourth cases of (4.55), we have either $\ddot{\gamma}(s_1) \succeq 1$ or $\ddot{\gamma}(s_1) \succeq 01$, in particular, $\ddot{f}_i^*(\boldsymbol{x}) \not\succeq 00$. This implies

$$\mathcal{P}^2_{\ddot{F},i} \subseteq \{01, 10, 11\}. \tag{4.316}$$

By $|\mathcal{P}^2_{F,i}| = 3$, there exists $\boldsymbol{c} = c_1c_2 \in \mathcal{C}^2$ such that

$$\mathcal{P}^2_{F,i} = \{c_1c_2, \bar{c_1}0, \bar{c_1}1\}. \tag{4.317}$$

Then there exists $\boldsymbol{x}' \in \mathcal{S}^+$ such that

$$f_i^*(\boldsymbol{x}') \succeq \boldsymbol{c}. \tag{4.318}$$

Let $\gamma(s'_1)\gamma(s'_2)\ldots\gamma(s'_{\rho'})$ be the γ-decomposition of $f_i(x'_1)$. Now we show $|\gamma(s'_1)| \geq 2$ by deriving a contradiction for the following two cases separately: the case $|\gamma(s'_1)| = 0$ and the case $|\gamma(s'_1)| = 1$.

– If we assume $|\gamma(s_1')| = 0$: We have

$$|\mathcal{P}^2_{F,i}| \overset{\text{(A)}}{\geq} |\bar{\mathcal{P}}^2_{F,i}| + |\mathcal{P}^2_{F,\tau_i(s_1')}| \tag{4.319}$$
$$\overset{\text{(B)}}{\geq} |\bar{\mathcal{P}}^0_{F,i}| + |\mathcal{P}^2_{F,\tau_i(s_1')}| \tag{4.320}$$
$$\overset{\text{(C)}}{\geq} 1 + |\mathcal{P}^2_{F,\tau_i(s_1')}| \tag{4.321}$$
$$\overset{\text{(D)}}{\geq} 1 + 3 \tag{4.322}$$
$$= 4, \tag{4.323}$$

where (A) follows from Lemma 2.2.3 and $|\gamma(s_1')| = 0$, (B) follows from Corollary 2.3.1 (ii) (b), (C) follows from Lemma 2.2.2 (iii) because f_i is injective by Lemma 4.2.13, and (D) follows from $F \in \mathscr{F}_2$. This conflicts with $|\mathcal{P}^2_{F,i}| = 3$.

– If we assume $|\gamma(s_1')| = 1$: We have

$$\mathcal{P}^2_{F,i} \overset{\text{(A)}}{\supseteq} \bar{\mathcal{P}}^2_{F,i} \tag{4.324}$$
$$\overset{\text{(B)}}{\supseteq} c_1 \mathcal{P}^1_{F,i}(c_1) \tag{4.325}$$
$$\overset{\text{(C)}}{=} c_1 \mathcal{P}^1_{F,i}(f_i(s_1')) \tag{4.326}$$
$$\overset{\text{(D)}}{\supseteq} c_1 \mathcal{P}^1_{F,\tau_i(s_1')} \tag{4.327}$$
$$\overset{\text{(E)}}{=} c_1\{0,1\} \tag{4.328}$$
$$\ni c_1\bar{c}_2, \tag{4.329}$$

where (A) follows from Lemma 2.2.1 (i), (B) follows from Lemma 2.2.1 (ii), (C) follows since $c_1 = f_i(s_1')$ by (4.318) and $|\gamma(s_1')| = 1$, (D) follows from Lemma 2.2.1 (i), and (E) follows from $F \in \mathscr{F}_2 \subseteq \mathscr{F}_1$. This conflicts with (4.317).

Hence, we have $|\gamma(s_1')| \geq 2$ and thus $\gamma(s_1') \succeq c_1c_2$ by (4.318). Therefore, by the third case of (4.55), we obtain $\ddot{f}_i^*(\boldsymbol{x}') \succeq \ddot{f}_i^*(x_1') \succeq \ddot{\gamma}(s_1') \succeq 01$, which leads to

$$01 \in \mathcal{P}^2_{\ddot{F},i}. \tag{4.330}$$

Next, we show that

$$10, 11 \in \mathcal{P}^2_{\ddot{F},i}. \tag{4.331}$$

To prove it, we choose $a \in \mathcal{C}$ arbitrarily and show that $1a \in \mathcal{P}^2_{\ddot{F},i}$. Since $\bar{c_1}a \in \mathcal{P}^2_{F,i}$ by (4.317), there exists $\boldsymbol{x}'' \in \mathcal{S}^+$ such that

$$f_i^*(\boldsymbol{x}'') \succeq \bar{c_1}a. \tag{4.332}$$

Let $\gamma(s''_1)\gamma(s''_2)\ldots\gamma(s''_{\rho''})$ be the γ-decomposition of $f_i(x''_1)$. We consider the following two cases separately: the case $|\gamma(s''_1)| \geq 2$ and the case $|\gamma(s''_1)| = 1$.

- The case $|\gamma(s''_1)| \geq 2$: Then we have $\gamma(s''_1) \succeq \bar{c_1}a$ by (4.332). Hence, by $|\mathcal{P}^2_{F,i}| = 3$, $|\gamma(s''_1)| \geq 2$, and (4.317), we have $\ddot{\gamma}(s''_1) \succeq 1a$ applying the fourth case of (4.55). Thus, we obtain $\ddot{f}_i^*(\boldsymbol{x}'') \succeq \ddot{\gamma}(s''_1) \succeq 1a$, which leads to $1a \in \mathcal{P}^2_{\ddot{F},i}$ as desired.
- The case $|\gamma(s''_1)| = 1$: We have

$$\begin{aligned}
\mathcal{P}^2_{\ddot{F},i} &\overset{\text{(A)}}{\supseteq} \bar{\mathcal{P}}^2_{\ddot{F},i} && (4.333)\\
&\overset{\text{(B)}}{\supseteq} 1\mathcal{P}^1_{\ddot{F},i}(1) && (4.334)\\
&\overset{\text{(C)}}{=} 1\mathcal{P}^1_{\ddot{F},i}(\ddot{\gamma}(s''_1)) && (4.335)\\
&\overset{\text{(D)}}{\supseteq} 1\mathcal{P}^1_{\ddot{F},\ddot{\tau}_i(s''_1)} && (4.336)\\
&\overset{\text{(E)}}{=} 1\{0,1\} && (4.337)\\
&\ni 1a, && (4.338)
\end{aligned}$$

  where (A) follows from Lemma 2.2.1 (i), (B) follows from Lemma 2.2.1 (ii), (C) is obtained by applying the second case of (4.55) by $|\mathcal{P}^2_{F,i}| = 3$ and $|\gamma(s''_1)| = 1$, (D) follows from Lemma 2.2.1 (i), and (E) follows from $F \in \mathscr{F}_2 \subseteq \mathscr{F}_1$.

Therefore, we conclude that (4.331) holds. By (4.316), (4.330), and (4.331), we obtain $\mathcal{P}^2_{\ddot{F},i} = \{01, 10, 11\}$ as desired.

(II) The case $|\mathcal{P}^2_{F,i}| = 4$: We consider the following two cases separately: (II-A) the case $\mathcal{S}_{F,i}(\lambda) \neq \emptyset$; (II-B) the case $\mathcal{S}_{F,i}(\lambda) = \emptyset$.

(II-A) The case $\mathcal{S}_{F,i}(\lambda) \neq \emptyset$: Since f_i is injective by Lemma 4.2.13, we can choose $s \in \mathcal{S}$ such that $\mathcal{S}_{F,i}(\lambda) = \{s\}$. Also, we have $\bar{\mathcal{P}}^0_{F,i} \neq \emptyset$

applying Lemma 2.2.2 (iii). Hence, by Lemma 4.2.2, we have $|\mathcal{P}^2_{F,\tau_i(s)}| \leq 3$. In particular, it holds that $|\mathcal{P}^2_{F,\tau_i(s)}| = 3$ by $F \in \mathscr{F}_2$. Therefore, by the result of the case (I), we obtain

$$\mathcal{P}^2_{\ddot{F},\tau_i(s)} = \{01, 10, 11\}. \tag{4.339}$$

Since f_i is injective, we can choose $s' \in \mathcal{S}$ such that $s' \neq \lambda$. Let $\gamma(s'_1)\gamma(s'_2)\dots\gamma(s'_{\rho'})$ be the γ-decomposition of $f_i(s')$. By Lemma 4.2.6 (i) and $\mathcal{S}_{F,i}(\lambda) \neq \emptyset$, we have

$$\gamma(s'_1) = \lambda. \tag{4.340}$$

Note that $\rho' \geq 2$ holds by (4.340) and $s'_{\rho'} = s' \neq \lambda$. We have

$$\begin{aligned} \ddot{f}_i(s') &= \ddot{\gamma}(s'_1)\ddot{\gamma}(s'_2)\dots\ddot{\gamma}(s'_{\rho'}) && (4.341)\\ &\succeq \ddot{\gamma}(s'_1)\ddot{\gamma}(s'_2) && (4.342)\\ &\overset{\text{(A)}}{=} \ddot{\gamma}(s'_2) && (4.343)\\ &\overset{\text{(B)}}{\succeq} 00, && (4.344) \end{aligned}$$

where (A) follows from (4.340) and Lemma 4.2.15 (i), and (B) follows from the fifth case of (4.55).
Hence, we have

$$00 \in \bar{\mathcal{P}}^2_{\ddot{F},i}. \tag{4.345}$$

We obtain

$$\mathcal{P}^2_{\ddot{F},i} \overset{\text{(A)}}{\supseteq} \mathcal{P}^2_{\ddot{F},\tau_i(s)} \cup \bar{\mathcal{P}}^2_{\ddot{F},i} \overset{\text{(B)}}{\supseteq} \{01, 10, 11\} \cup \{00\} = \{00, 01, 10, 11\} \tag{4.346}$$

as desired, where (A) follows from Lemma 2.2.1 (i), and (B) follows from (4.339) and (4.345).

(II-B) The case $\mathcal{S}_{F,i}(\lambda) = \emptyset$: It suffices to show that $\mathcal{P}^2_{\ddot{F},i} \supseteq \mathcal{P}^2_{F,i}$ since $|\mathcal{P}^2_{F,i}| = 4$. Choose $\boldsymbol{c} = c_1c_2 \in \mathcal{P}^2_{F,i} = \{00, 01, 10, 11\}$ arbitrarily. Then there exists $\boldsymbol{x} \in \mathcal{S}^+$ such that

$$f^*_i(\boldsymbol{x}) \succeq \boldsymbol{c}. \tag{4.347}$$

Let $\gamma(s_1)\gamma(s_2)\dots\gamma(s_\rho)$ be the γ-decomposition of $f_i(x_1)$. We consider the following two cases separately: the case $|\gamma(s_1)| \geq 2$ and the case $|\gamma(s_1)| = 1$. Note that we can exclude the case $|\gamma(s_1)| = 0$ since $\mathcal{S}_{F,i}(\lambda) = \emptyset$.

* The case $|\gamma(s_1)| \geq 2$: We have

$$\ddot{f}_i(x_1) \succeq \ddot{\gamma}(s_1) \overset{\text{(A)}}{=} \gamma(s_1) \overset{\text{(B)}}{\succeq} \boldsymbol{c}, \tag{4.348}$$

where (A) follows from $|\mathcal{P}^2_{F,i}| = 4$ and the first case of (4.55), and (B) follows from (4.347) and $|\gamma(s_1)| \geq 2$. This implies $\boldsymbol{c} \in \mathcal{P}^2_{\ddot{F},i}$ as desired.

* The case $|\gamma(s_1)| = 1$: We have

$$\ddot{f}_i(s_1) = \ddot{\gamma}(s_1) \overset{\text{(A)}}{=} \gamma(s_1) \overset{\text{(B)}}{=} c_1, \tag{4.349}$$

where (A) follows from $|\mathcal{P}^2_{F,i}| = 4$ and the first case of (4.55), and (B) follows from (4.347) and $|\gamma(s_1)| = 1$.

Put $j \coloneqq \tau_i(s_1)$. By Lemma 2.3.3, we can choose the longest sequence $\boldsymbol{x}' \in \mathcal{S}^+$ such that $f^*_j(\boldsymbol{x}') = \lambda$. Then we have $\mathcal{S}_{F,\tau^*_j(\boldsymbol{x}')}(\lambda) = \emptyset$. Also, we have $|\mathcal{P}^2_{F,\tau^*_j(\boldsymbol{x}')}| \geq 3$ by $F \in \mathscr{F}_2$. In particular, we have one of the following conditions (a) and (b).

(a) $|\mathcal{P}^2_{F,\tau^*_j(\boldsymbol{x}')}| = 3$.

(b) $|\mathcal{P}^2_{F,\tau^*_j(\boldsymbol{x}')}| = 4$ and $\mathcal{S}_{F,\tau^*_j(\boldsymbol{x}')}(\lambda) = \emptyset$.

Therefore, from the cases (I) and (II-A) proven above, we have $\mathcal{P}^2_{\ddot{F},\ddot{\tau}^*_j(\boldsymbol{x}')} \supseteq \{01, 10, 11\}$, which leads to

$$\mathcal{P}^1_{\ddot{F},\ddot{\tau}^*_j(\boldsymbol{x}')} = \{0, 1\} \tag{4.350}$$

by Lemma 2.3.2 (i). Thus, we have

$$\begin{aligned}
&\mathcal{P}^2_{\ddot{F},i} \\
&\overset{\text{(A)}}{\supseteq} \bar{\mathcal{P}}^2_{\ddot{F},i} \overset{\text{(B)}}{\supseteq} c_1\mathcal{P}^1_{\ddot{F},i}(c_1) \overset{\text{(C)}}{=} c_1\mathcal{P}^1_{\ddot{F},i}(\ddot{f}_i(s_1)) \\
&\overset{\text{(D)}}{\supseteq} c_1\mathcal{P}^1_{\ddot{F},j} \overset{\text{(D)}}{\supseteq} c_1\mathcal{P}^1_{\ddot{F},\ddot{\tau}^*_j(x'_1)} \overset{\text{(D)}}{\supseteq} c_1\mathcal{P}^1_{\ddot{F},\ddot{\tau}^*_j(x'_1x'_2)} \overset{\text{(D)}}{\supseteq} \cdots \overset{\text{(D)}}{\supseteq} c_1\mathcal{P}^1_{\ddot{F},\ddot{\tau}^*_j(\boldsymbol{x}')} \\
&\overset{\text{(E)}}{=} c_1\{0, 1\} \ni c_1c_2 = \boldsymbol{c},
\end{aligned} \tag{4.351}$$

where (A) follows from Lemma 2.2.1 (i), (B) follows from Lemma 2.2.1 (ii), (C) follows from (4.349), (D)s follow from Lemma 2.2.1 (i), and (E) follows from (4.350). Therefore, we conclude that $\mathcal{P}^2_{\ddot{F},i} \supseteq \mathcal{P}^2_{F,i} = \{00, 01, 10, 11\}$ as desired.

(Proof of (ii)): We have

$$
\begin{aligned}
&\bar{\mathcal{P}}^0_{F,i}(f_i(s)) \neq \emptyset \\
&\overset{\text{(A)}}{\Longleftrightarrow} \quad \bar{\mathcal{P}}^2_{F,i}(f_i(s)) \neq \emptyset && (4.352)\\
&\Longleftrightarrow \quad {}^{\exists}\boldsymbol{x} \in \mathcal{S}^+; {}^{\exists}\boldsymbol{c} \in \mathcal{C}^2; (f_i^*(\boldsymbol{x}) \succeq f_i(s)\boldsymbol{c}, f_i(x_1) \succ f_i(s)) && (4.353)\\
&\overset{\text{(B)}}{\Longleftrightarrow} \quad {}^{\exists}\boldsymbol{x} \in \mathcal{S}^+; {}^{\exists}\boldsymbol{c} \in \mathcal{C}^2; (\ddot{f}_i^*(\boldsymbol{x}) \succeq \ddot{f}_i(s)\boldsymbol{c}, \ddot{f}_i(x_1) \succ \ddot{f}_i(s)) && (4.354)\\
&\Longleftrightarrow \quad \bar{\mathcal{P}}^2_{\ddot{F},i}(\ddot{f}_i(s)) \neq \emptyset, && (4.355)
\end{aligned}
$$

where (A) follows from Corollary 2.3.1 (ii) (a), and (B) follows from Lemma 4.2.15 (iii).

We consider the following two cases separately: the case $\bar{\mathcal{P}}^0_{F,i}(f_i(s)) = \emptyset$ and the case $\bar{\mathcal{P}}^0_{F,i}(f_i(s)) \neq \emptyset$.

- The case $\bar{\mathcal{P}}^0_{F,i}(f_i(s)) = \emptyset$: By (4.355), the condition $\bar{\mathcal{P}}^0_{F,i}(f_i(s)) = \emptyset$ is equivalent to $\bar{\mathcal{P}}^2_{\ddot{F},i}(\ddot{f}_i(s)) = \emptyset$ as desired.
- The case $\bar{\mathcal{P}}^0_{F,i}(f_i(s)) \neq \emptyset$: Then since $\bar{\mathcal{P}}^2_{\ddot{F},i}(\ddot{f}_i(s)) \neq \emptyset$ holds by (4.355), it suffices to show that $\bar{\mathcal{P}}^2_{\ddot{F},i}(\ddot{f}_i(s)) \subseteq \{00\}$. Moreover, to prove this, it suffices to show that for any $\boldsymbol{x} \in \mathcal{S}^+$ such that $\ddot{f}_i(x_1) \succ \ddot{f}_i(s)$, we have $\ddot{f}_i^*(\boldsymbol{x}) \succeq \ddot{f}_i(s)00$.

  Choose $\boldsymbol{x} \in \mathcal{S}^+$ such that

  $$\ddot{f}_i(x_1) \succ \ddot{f}_i(s). \tag{4.356}$$

  Let $\gamma(s_1)\gamma(s_2)\ldots\gamma(s_\rho)$ be the γ-decomposition of $f_i(x_1)$. Because $f_i(x_1) \succ f_i(s)$ holds by (4.356) and Lemma 4.2.15 (iii), we have $s = s_r$ and $\ddot{f}_i(s) = \ddot{\gamma}(s_1)\ddot{\gamma}(s_2)\ldots\ddot{\gamma}(s_r)$ for some $r = 1, 2, \ldots, \rho - 1$. For such r, we have

  $$
  \begin{aligned}
  \ddot{f}_i^*(\boldsymbol{x}) &\succeq \ddot{f}_i(x_1) && (4.357)\\
  &= \ddot{\gamma}(s_1)\ddot{\gamma}(s_2)\ldots\ddot{\gamma}(s_r)\ddot{\gamma}(s_{r+1})\ldots\ddot{\gamma}(s_\rho) && (4.358)\\
  &\succeq \ddot{f}_i(s)\ddot{\gamma}(s_{r+1}) && (4.359)\\
  &\overset{\text{(A)}}{\succeq} \ddot{f}_i(s)00 && (4.360)
  \end{aligned}
  $$

  as desired, where (A) follows from the fifth case of (4.55).

□

Chapter 5

Conclusion

We considered a general class of source codes which allow a finite number of code tables and at most k-bit decoding delay for $k \geq 0$.

In Chapter 2, we first formalized source codes with a finite number of code tables as code-tuples in Section 2.1, and we stated two equivalent definitions of the class $\mathscr{F}_{k\text{-dec}}$ of k-bit delay decodable code-tuples in Section 2.2. To exclude some abnormal code-tuples from consideration, in Section 2.3 we introduced the class $\mathscr{F}_{\mathrm{ext}}$ of extendable code-tuples, which are code-tuples F with $\mathcal{P}^1_{F,i} \neq \emptyset$ for any $i \in [F]$. In Section 2.4, we defined the average codeword length $L(F)$ of a code-tuple F based on a stationary distribution of the Markov process induced by F, and we limited the scope of consideration to the class $\mathscr{F}_{\mathrm{reg}}$ of regular code-tuples, which have a unique stationary distribution. Then in Section 2.5, we defined the class $\mathscr{F}_{\mathrm{irr}}$ of irreducible code-tuples and introduced irreducible parts of a code-tuple F, which are obtained by removing the transient code tables from F.

In Chapter 3, for a fixed source distribution μ, we investigated the general properties of *k-bit delay optimal* code-tuples, which are code-tuples with the optimal average codeword length in $\mathscr{F}_{\mathrm{reg}} \cap \mathscr{F}_{\mathrm{ext}} \cap \mathscr{F}_{k\text{-dec}}$. Then we proved three theorems as part of the main results.

- The first Theorem 3.1.1 claims the existence of an irreducible k-bit delay optimal code-tuple F such that $\mathcal{P}^k_{F,0}, \mathcal{P}^k_{F,1}, \ldots, \mathcal{P}^k_{F,|F|-1}$ are distinct. This result gives an upper bound of the required number of code tables for k-bit delay optimal code-tuples. Using this theorem, we proved the existence of a k-bit delay optimal code-tuple, that is, it is not the case that one can achieve an arbitrarily small average codeword length by using arbitrarily many code tables.

- The second Theorem 3.1.2 states that for a k-bit delay optimal code-tuple $F(f, \tau)$, if the first k bits of a given $\boldsymbol{b} \in \mathcal{C}^*$ is a prefix of $f_i^*(\boldsymbol{x})$ of some $\boldsymbol{x} \in \mathcal{S}^*$, then $\boldsymbol{b}$ itself is also a prefix of $f_i^*(\boldsymbol{x}')$ of some $\boldsymbol{x}' \in \mathcal{S}^*$. This result is a generalization of the property of Huffman codes that each internal node in the code tree has two child nodes.

- The third Theorem 3.1.3 guarantees the existence of a k-bit delay optimal code-tuple in the class $\mathscr{F}_{\text{fork}}$ of the code-tuples F such that $\mathcal{P}^1_{F,i} = \{0, 1\}$ for any $i \in [F]$. Therefore, it is sufficient to consider only the code-tuples F such that both $0, 1 \in \mathcal{C}$ are possible as the first bit of codeword no matter which code table of F we start the encoding process from.

These theorems enable us to limit the scope of codes to be considered when discussing k-bit delay optimal code-tuples.

In Section 4, as applications of the three theorems, for $k = 1, 2$, we gave a class of code-tuples which can achieve the optimal average codeword length in $\mathscr{F}_{\text{reg}} \cap \mathscr{F}_{\text{ext}} \cap \mathscr{F}_{k\text{-dec}}$ for a given source distribution μ.

- In Section 4.1, we proved Theorem 4.1.1 that the Huffman code achieves the optimal average codeword length in $\mathscr{F}_{\text{reg}} \cap \mathscr{F}_{\text{ext}} \cap \mathscr{F}_{1\text{-dec}}$. Namely, the class of instantaneous codes with a single code table can achieve the optimal average codeword length in $\mathscr{F}_{\text{reg}} \cap \mathscr{F}_{\text{ext}} \cap \mathscr{F}_{1\text{-dec}}$.

- In Section 4.2, we proved Theorem 4.2.1 that $\mathscr{F}_{2\text{-opt}} \cap \mathscr{F}_{\text{AIFV}} \neq \emptyset$, that is, there exists an AIFV code which is 2-bit delay optimal, and thus the class of AIFV code can achieve the optimal average codeword length in $\mathscr{F}_{\text{reg}} \cap \mathscr{F}_{\text{ext}} \cap \mathscr{F}_{2\text{-dec}}$.

Finally, we describe future works below.

- Finding a good optimal subclass of $\mathscr{F}_{\text{reg}} \cap \mathscr{F}_{\text{ext}} \cap \mathscr{F}_{k\text{-dec}}$: In Chapter 4, we presented subclasses of $\mathscr{F}_{\text{reg}} \cap \mathscr{F}_{\text{ext}} \cap \mathscr{F}_{k\text{-dec}}$ which can achieve the optimal average codeword length in $\mathscr{F}_{\text{reg}} \cap \mathscr{F}_{\text{ext}} \cap \mathscr{F}_{k\text{-dec}}$ for $k = 1, 2$. What subclasses could be considered for the case $k \geq 3$?

- Extending to *alphabetic* codes: An alphabetic code is a source code $f^* : \mathcal{S}^* \to \mathcal{C}^*$ which preserves the lexicographical order of total orders over $\mathcal{S}$ and $\mathcal{C}$ in the encoding process, that is, satisfies

$$^{\forall}\boldsymbol{x}, \boldsymbol{y} \in \mathcal{S}^*; (\boldsymbol{x} \leq^*_{\mathcal{S}} \boldsymbol{y} \implies f^*(\boldsymbol{x}) \leq^*_{\mathcal{C}} f^*(\boldsymbol{y})), \tag{5.1}$$

where $(\mathcal{S}, \leq_\mathcal{S})$ and $(\mathcal{C}, \leq_\mathcal{C})$ are totally ordered sets, and $\leq_\mathcal{S}^*$ and $\leq_\mathcal{C}^*$ denote the lexicographical orders over $\mathcal{S}^*$ and $\mathcal{C}^*$ corresponding to $\leq_\mathcal{S}$ and $\leq_\mathcal{C}$, respectively. What results will be obtained if such constraint is imposed on our discussion?

- Generalizing to d-ary coding: How can we generalize our results to d-ary coding, in which a source sequence $\boldsymbol{x} \in \mathcal{S}^*$ is encoded to a codeword sequence over the d-ary coding alphabet $\mathcal{C} \coloneqq \{0, 1, 2, \ldots, d-1\}$ instead of $\{0, 1\}$ for a general integer $d \geq 2$?

Appendix A

List of Notations

$\mathcal{A} \times \mathcal{B}$	the Cartesian product of sets $\mathcal{A}$ and $\mathcal{B}$, that is, $\{(a, b) : a \in \mathcal{A}, b \in \mathcal{B}\}$, defined at the beginning of Section 2.
$\|\mathcal{A}\|$	the cardinality of a set $\mathcal{A}$, defined at the beginning of Section 2.
$\mathcal{A}^k$	the set of all sequences of length k over a set $\mathcal{A}$, defined at the beginning of Section 2.
$\mathcal{A}^{\geq k}$	the set of all sequences of length greater than or equal to k over a set $\mathcal{A}$, defined at the beginning of Section 2.
$\mathcal{A}^{\leq k}$	the set of all sequences of length less than or equal to k over a set $\mathcal{A}$, defined at the beginning of Section 2.
$\mathcal{A}^*$	the set of all sequences of finite length over a set $\mathcal{A}$, defined at the beginning of Section 2.
$\mathcal{A}^+$	the set of all sequences of finite positive length over a set $\mathcal{A}$, defined at the beginning of Section 2.
$a_{F,i}$	defined in Definition 4.2.4.
$\mathcal{C}$	the coding alphabet $\mathcal{C} = \{0, 1\}$, at the beginning of Section 2.
$\bar{c}$	the negation of $c \in \mathcal{C}$, that is, $\bar{0} = 1, \bar{1} = 0$ defined at the beginning of the proof of Theorem 3.1.2.
$\boldsymbol{c}\mathcal{A}$	$\{\boldsymbol{cb} : \boldsymbol{b} \in \mathcal{A}\}$ for $\boldsymbol{c} \in \mathcal{C}^*$ and $\mathcal{A} \subseteq \mathcal{C}^*$, defined at the beginning of Section 2.
$[\boldsymbol{c}]_k$	the prefix of length k of $\boldsymbol{c} \in \mathcal{C}^{\geq k}$, defined at the beginning of Section 2.
$d_{F,i}$	defined in (3.123).
f_i^*	defined in Definition 2.1.3.

F	simplified notation of a code-tuple $F(f_0, f_1, \ldots, f_{m-1}, \tau_0, \tau_1, \ldots, \tau_{m-1})$, also written as $F(f, \tau)$, defined below Definition 2.1.1.
$\bar{F}$	an irreducible part of F, defined in Definition 2.5.4.
$\|F\|$	the number of code tables of F, defined below Definition 2.1.1.
$[F]$	simplified notation of $[\|F\|] = \{0, 1, 2, \ldots, \|F\| - 1\}$, defined below Definition 2.1.1.
$\widehat{F}$	defined in Definition 3.4.1.
$\dot{F}$	defined in Definition 4.2.4.
$\ddot{F}$	defined in Definition 4.2.5.
$\mathscr{F}_{\mathrm{AIFV}}$	the set of all AIFV codes, defined in Definition 4.2.1.
$\mathscr{F}^{(m)}$	the set of all m-code-tuples, defined after Definition 2.1.1.
$\mathscr{F}$	the set of all code-tuples, defined after Definition 2.1.1.
$\mathscr{F}_{\mathrm{ext}}$	the set of all extendable code-tuples, defined in Definition 2.3.1.
$\mathscr{F}_{\mathrm{fork}}$	defined in Definition 3.1.3.
$\mathscr{F}_{k\text{-opt}}$	the set of all k-bit delay optimal code-tuples, defined in Definition 3.1.1.
$\mathscr{F}_{\mathrm{reg}}$	the set of all regular code-tuples, defined in Definition 2.4.3.
$\mathscr{F}_0$	$\{F \in \mathscr{F}_{\mathrm{reg}} \cap \mathscr{F}_{\text{2-dec}} : {}^{\forall}i \in [F]; \mathcal{P}^1_{F,i} \neq \emptyset\} = \mathscr{F}_{\mathrm{reg}} \cap \mathscr{F}_{\mathrm{ext}} \cap \mathscr{F}_{\text{2-dec}}$, defined in Definition 4.2.2.
$\mathscr{F}_1$	$\{F \in \mathscr{F}_{\mathrm{reg}} \cap \mathscr{F}_{\text{2-dec}} : {}^{\forall}i \in [F]; \mathcal{P}^1_{F,i} = \{0, 1\}\}$, defined in Definition 4.2.2.
$\mathscr{F}_2$	$\{F \in \mathscr{F}_{\mathrm{reg}} \cap \mathscr{F}_{\text{2-dec}} : {}^{\forall}i \in [F]; \|\mathcal{P}^2_{F,i}\| \geq 3\}$, defined in Definition 4.2.2.
$\mathscr{F}_3$	$\{F \in \mathscr{F}_{\mathrm{reg}} \cap \mathscr{F}_{\text{2-dec}} : {}^{\forall}i \in [F]; \mathcal{P}^2_{F,i} \supseteq \{01, 10, 11\}\}$, defined in Definition 4.2.2.
$\mathscr{F}_4$	$\{F \in \mathscr{F}_{\mathrm{reg}} \cap \mathscr{F}_{\text{2-dec}} \cap \mathscr{F}^{(2)} : \mathcal{P}^2_{F,0} = \{00, 01, 10, 11\}, \mathcal{P}^2_{F,1} = \{01, 10, 11\}\}$, defined in Definition 4.2.2.
$h(F)$	defined after Lemma 3.2.3.
$l_{F,i}$	defined in (3.153).
$L(F)$	the average codeword length of a code-tuple F, defined in Definition 2.4.4.
$L_i(F)$	the average codeword length of the i-th code table of F, defined in Definition 2.4.4.
$[m]$	$\{0, 1, 2, \ldots, m - 1\}$, defined at the beginning of Section 2.
$\mathcal{M}_F$	$\{i \in [F] : \|\mathcal{P}^2_{F,i}\| = 2\}$, defined in Lemma 4.2.11.
$\mathcal{P}^k_{F,i}$	defined in Definition 2.2.1.

$\bar{\mathcal{P}}^{k}_{F,i}$	defined in Definition 2.2.1.
$\mathcal{P}^{*}_{F,i}$	defined in Definition 2.2.2.
$\bar{\mathcal{P}}^{*}_{F,i}$	defined in Definition 2.2.2.
$\mathscr{P}^{k}_{F}$	defined in Definition 3.1.2.
$\mathrm{pref}(\boldsymbol{x})$	the sequence obtained by deleting the last letter of $\boldsymbol{x}$, defined at the beginning of Section 2.
$Q(F)$	the transition probability matrix, defined in Definition 2.4.1.
$Q_{i,j}(F)$	the transition probability, defined in Definition 2.4.1.
$\mathbb{R}$	the set of all real numbers.
$\mathbb{R}^m$	the set of all m-dimensional real row vectors for an integer $m \geq 1$.
$\mathcal{S}$	the source alphabet, defined at the beginning of Section 2.
$\mathrm{suff}(\boldsymbol{x})$	the sequence obtained by deleting the first letter of $\boldsymbol{x}$, defined at the beginning of Section 2.
x_i	the i-th letter of a sequence $\boldsymbol{x}$, defined at the beginning of Section 2.
$\boldsymbol{x} \wedge \boldsymbol{y}$	the longest common prefix of $\boldsymbol{x}$ and $\boldsymbol{y}$, defined after Theorem 3.1.3.
$\boldsymbol{x} \preceq \boldsymbol{y}$	$\mathbf{x}$ is a prefix of $\mathbf{y}$, defined at the beginning of Section 2.
$\boldsymbol{x} \prec \boldsymbol{y}$	$\boldsymbol{x} \preceq \boldsymbol{y}$ and $\boldsymbol{x} \neq \boldsymbol{y}$, defined at the beginning of Section 2.
$\boldsymbol{x} \not\succeq\not\preceq \boldsymbol{y}$	$\boldsymbol{x}$ is not a prefix of $\boldsymbol{y}$, and $\boldsymbol{y}$ is not a prefix of $\boldsymbol{x}$, defined after Theorem 3.1.3.
$\lvert\boldsymbol{x}\rvert$	the length of a sequence $\boldsymbol{x}$, defined at the beginning of Section 2.
$\boldsymbol{x}^{-1}\boldsymbol{y}$	the sequence $\boldsymbol{z}$ such that $\boldsymbol{x}\boldsymbol{z} = \boldsymbol{y}$, defined at the beginning of Section 2.
$\gamma(s_r)$	defined in Definition 4.2.3.
λ	the empty sequence, defined at the beginning of Section 2.
$\mu(s)$	the probability of occurrence of symbol s, defined at the beginning of Subsection 2.4.
$\boldsymbol{\pi}(F)$	defined in Definition 2.4.3.
τ^{*}_{i}	defined in Definition 2.1.3.

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