Algorithms for approximating finite Hilbert transform with end-point singularities and its derivatives

Algorithms for approximating finite Hilbert transform with end-point singularities and its derivatives

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Abstract

Algorithms are proposed for the numerical evaluation of Cauchy principal value integrals $⨍_{-1}^{1} w(t)f(t)/(t-x)dt$, $-1 < x < 1$, with weight functions of Jacobi type singularities $w(t) = (1-t)^{\alpha}(1+t)^{\beta}$, where $\alpha = \pm 1/2$ and $\beta = \pm 1/2$, for a given function $f(t)$ and Hadamard finite-part integrals $⨎_{-1}^{1} w(t)f(t)/(t-x)^2 dt$. The function f is interpolated by using a finite sum of Chebyshev polynomials. The present algorithms require $O(N \log N)$ arithmetic operations, where N is the order of the interpolation polynomial. It is shown that the present scheme gives uniform approximations, namely the errors are bounded independently of x, and is very efficient for smooth f. Further, we discuss approximations of hyper-singular integrals $\int_{-1}^{1} w(t)f(t)/(t-x)^n dt$, $n \geq 3$, and show their uniform convergences. Numerical examples are given to demonstrate the performance of the present schemes.

Keywords: quadrature rule, Hilbert transform, principal value integral, finite-part integral, endpoint singularities of Jacobi type, Chebyshev interpolation, error analysis, uniform approximation, three-term recurrence relations

1. Introduction

Let $Q^{(n,i)}(f;x)$ ($n = 0, 1, \ldots$, $i = 1, \ldots, 4$) be generalized finite Hilbert transforms of a given function $f(t)$ with weight functions of Jacobi type end-point singularities $(1-t)^{\alpha}(1+t)^{\beta}$ [7], $\alpha = \pm 1/2$, $\beta = \pm 1/2$, defined by

$$Q^{(n,i)}(f;x) \quad = \quad ⨍_{-1}^{1} \frac{w_i(t)f(t)}{(t-x)^{n+1}}dt = \frac{1}{n!}\frac{d^n}{dx^n}⨍_{-1}^{1} \frac{w_i(t)f(t)}{t-x}dt$$

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$$=: \quad \frac{1}{n!}\frac{d^n}{dx^n} I^{(i)}(f;x), \quad -1 < x < 1,\ i = 1, \ldots, 4, \tag{1}$$

where

$$w_1(t) = \frac{1}{w_2(t)} = \frac{1}{\sqrt{1-t^2}}, \quad w_3(t) = \frac{1}{w_4(t)} = \sqrt{\frac{1+t}{1-t}},$$

are also weight functions for the orthogonalities of Chebyshev polynomials of the first to fourth kind, respectively [13, p.73].

The evaluation of $Q^{(n,i)}(f;x)$ (1) is required in solving many problems of science and engineering. A solution of the following integral equation appearing in physics and engineering,

$$\frac{1}{\pi} ⨍_{-1}^{1} \frac{g(t)}{t-x} dt = f(x),$$

under the condition that $\int_{-1}^{1} g(t)dt = c$, where c is a specified constant, is given by $g(t) = w_2(t)\{c - I^{(2)}(f;t)\}/\pi$, see Srivastav [17], where

$$I^{(i)}(f;x) = Q^{(0,i)}(f;x) = ⨍_{-1}^{1} \frac{w_i(t)f(t)}{t-x} dt, \quad -1 < x < 1, \tag{2}$$

is called Cauchy principal value integral [10, 11, 15].

Each $I^{(i)}(f;x)$ $(i = 2, 3, 4)$ can be obtained from $I^{(1)}(f;x)$. Indeed, since $w_2(t)/w_1(t) = 1 - x^2 - (t^2 - x^2)$, $w_3(t)/w_1(t) = 1 + x + (t - x)$ and $w_4(t)/w_1(t) = 1 - x - (t - x)$ we see from (2) that

$$\begin{aligned}
I^{(2)}(f;x) &= (1-x^2)I^{(1)}(f;x) - \int_{-1}^{1} w_1(t)(t+x)f(t)dt. &(3)\\
I^{(3)}(f;x) &= (1+x)I^{(1)}(f;x) + \int_{-1}^{1} w_1(t)f(t)dt, &(4)\\
I^{(4)}(f;x) &= (1-x)I^{(1)}(f;x) - \int_{-1}^{1} w_1(t)f(t)dt. &(5)
\end{aligned}$$

On the other hand, $Q^{(1,i)}(f;x)$ is called Hadamard finite-part integrals [16], which we denote as $J^{(i)}(f;x)$,

$$J^{(i)}(f;x) := Q^{(1,i)}(f;x) = \frac{dI^{(i)}(f;x)}{dx} = \not\!\!\int_{-1}^{1} \frac{w_i(t)f(t)}{(t-x)^2} dt, \ -1 < x < 1. \tag{6}$$

From (3), (4), (5) and (6) it follows that

$$\begin{aligned}
J^{(2)}(f;x) &= (1-x^2)J^{(1)}(f;x) - 2xI^{(1)}(f;x) - \int_{-1}^{1} w_1(t)f(t)dt, &(7)\\
J^{(3)}(f;x) &= (1+x)J^{(1)}(f;x) + I^{(1)}(f;x), &(8)\\
J^{(4)}(f;x) &= (1-x)J^{(1)}(f;x) - I^{(1)}(f;x). &(9)
\end{aligned}$$

The purpose of this paper is to present efficient quadrature methods for uniformly approximating $Q^{(n,i)}(f;x)$, particularly, $Q^{(0,i)}(f;x) = I^{(i)}(f;x)$ and $Q^{(1,i)}(f;x) = J^{(i)}(f;x)$ $(i = 1,\ldots,4)$ given by (2) and (6), respectively. We extend schemes due to Hasegawa [9] and Hasegawa and Torii [10, 11] for approximating $⨍_{-1}^{1} f(t)/(t-x)dt$ with no end-point singularities. The present scheme is of interpolatory integration [4, p.74], namely f is interpolated by an polynomial p_N and $Q^{(n,i)}(f;x)$ are approximated by $Q^{(n,i)}(p_N;x)$. In particular, f is approximated by a finite sum of the Chebyshev polynomials [2, 13], see (12) below. The present methods are simple to implement as well as numerically stable to compute [10] and very efficient for smooth functions f [19]. Further, we discuss the uniform convergences of the approximations $Q^{(n,i)}(p_N;x)$ to hyper-singular integrals $Q^{(n,i)}(f;x)$ $(n \geq 2,\ 1 \leq i \leq 4)$ [3, 14].

This paper is organized as follows. In Section 2 we give approximation methods for the Cauchy principal value integrals $I^{(i)}(f;x)$ and the Hadamard finite-part integrals $J^{(i)}(f;x)$. It is shown that the approximations for many values of $x \in (-1,1)$ are efficiently computed by using three-term recurrence relations, see (19) and (23). In Section 3 we show that the errors of the approximations to $I^{(i)}(f;x)$ and $j^{(i)}(f;x)$ are uniformly bounded, namely independently of the values of x and converge to 0 rapidly for smooth functions. Furthermore we discuss the errors of approximations to hyper-singular integrals $Q^{(n,i)}(f;x)$ $(n \geq 2)$ and show their uniform convergences. In Section 4 we give numerical examples to demonstrate the efficiency of the present schemes.

2. Approximation

The following relation, see (2.4) in Srivastav [17],

$$⨍_{-1}^{1} \frac{w_1(t)}{(t-x)^{n+1}} dt = \frac{1}{n!}\frac{d^n}{dx^n} ⨍_{-1}^{1} \frac{w_1(t)}{t-x} dt = 0, \quad n = 0,1,\ldots, \tag{10}$$

will be often used in constructing approximation methods for $I^{(i)}(f;x)$ and $J^{(i)}(f;x)$ and in their error analysis, as shown below. Further, we will use the orthogonality relation of the Chebyshev polynomials of the first kind $T_k(t)$, where $T_k(t) = \cos k\theta$, $t = \cos\theta$,

$$\int_{-1}^{1} w_1(t)\, T_j(t)\, T_k(t)\, dt = \alpha_k \delta_{j,k}, \tag{11}$$

where $\alpha_0 = \pi$ and $\alpha_k = \pi/2$ $(k \neq 0)$ and $\delta_{j,k} = 1$ if $j = k$, otherwise $\delta_{j,k} = 0$.

2.1. Interpolation

We approximate $f(t)$ in (2) by a finite sum of Chebyshev polynomials $T_k(t)$ given by

$$p_N(t) := {\sum_{k=0}^{N}}'' a_k^N T_k(t), \tag{12}$$

where the double prime denotes the summation whose first and last terms are halved. Let $W_{N+1}(t)$ be defined by

$$W_{N+1}(t) = T_{N+1}(t) - T_{N-1}(t). \tag{13}$$

The coefficients a_k^N in (12) are determined so that $p_N(t)$ interpolates $f(t)$ at the zeros of $W_{N+1}(t)$, namely $t_j = \cos \pi j/N$, $0 \le j \le N$, and are given by

$$a_k^N = \frac{2}{N} \sum_{j=0}^{N}{}'' f(\cos \pi j/N) \cos(\pi jk/N), \quad 0 \le k \le N. \tag{14}$$

The right-hand side of (14) can be efficiently evaluated by using the FFT(Fast Fourier Transform) [8, 20]. The approximation p_N (12) is of fast convergence for smooth function f.

2.2. Approximation to Cauchy principal value integrals

By using $p_N(t)$ (12) we obtain an approximation $I_N^{(1)}(f;x)$ to the integral $I^{(1)}(f;x)$ (2) as follows

$$I^{(1)}(f;x) \approx I_N^{(1)}(f;x) := I^{(1)}(p_N;x) = ⨍_{-1}^{1} \frac{w_1(t)p_N(t)}{t-x} dt. \tag{15}$$

The integral in the rightmost-hand of (15) can be easily evaluated in the following way. From (10) we have

$$I_N^{(1)}(f;x) = ⨍_{-1}^{1} \frac{w_1(t)\{p_N(t) - p_N(x)\}}{t-x} dt. \tag{16}$$

Expanding the integrand in (16) in terms of the Chebyshev polynomials,

$$\frac{p_N(t) - p_N(x)}{t-x} = \sum_{k=0}^{N-1}{}' b_k T_k(t) =: q_{N-1}(t), \tag{17}$$

and integrating term by term yield an integration formula by noting (11)

$$I_N^{(1)}(f;x) = \pi b_0/2. \tag{18}$$

The prime in (17) denotes the summation whose first term is halved. The coefficients b_k in (17) can be stably computed by using the recurrence relation

$$b_{k+1} - 2xb_k + b_{k-1} = 2a_k^N, \quad k = N, N-1, \ldots, 1, \tag{19}$$

in the backward direction with the starting values $b_N = b_{N+1} = 0$, where we take $a_N^N/2$ instead of a_N^N [10]. We have omitted the dependence of b_k on N and x.

Similarly we obtain approximations for $j = 0$ or 1

$$\int_{-1}^{1} w_1(t)t^j f(t)dt \approx \int_{-1}^{1} w_1(t)t^j p_N(t)dt = \frac{\pi}{2}a_j^N, \quad j = 0, 1, \tag{20}$$

by using the orthogonality relation (11) and noting that $t = T_1(t)$. From (3), (4), (5), (18), (20), we have integration formulas $I_N^{(i)}(f;x) = I^{(i)}(p_N;x)$ for $I^{(i)}(f;x)$ $(i = 2, 3, 4)$ as follows

$$\begin{aligned} I_N^{(2)}(f;x) &= \pi(1-x^2)b_0/2 - \pi(xa_0^N + a_1^N)/2, \\ I_N^{(3)}(f;x) &= \pi(1+x)b_0/2 + \pi a_0^N/2, \\ I_N^{(4)}(f;x) &= \pi(1-x)b_0/2 - \pi a_0^N/2. \end{aligned}$$

2.3. Approximation to Hadamard finite-part integrals

We approximate Hadamard finite-part integrals $J^{(i)}(f;x)$ (6) by $J_N^{(i)}(f;x) = J^{(i)}(p_N;x)$. Now, particularly we evaluate $J^{(1)}(p_N;x)$. From (10) we have

$$\begin{aligned} J^{(1)}(p_N;x) &= \int_{-1}^{1} \frac{w_1(t)\{p_N(t) - p_N(x)\}}{(t-x)^2}dt = \int_{-1}^{1} \frac{w_1(t)q_{N-1}(x)}{t-x}dt \\ &= \int_{-1}^{1} \frac{w_1(t)\{q_{N-1}(t) - q_{N-1}(x)\}}{t-x}dt. \end{aligned} \tag{21}$$

Expanding the integrand in (21) in terms of the Chebyshev polynomials,

$$\frac{q_{N-1}(t) - q_{N-1}(x)}{t-x} = {\sum_{k=0}^{N-2}}' c_k T_k(t), \tag{22}$$

and integrating term by term yield an integration formula

$$J_N^{(1)}(f;x) = \pi c_0/2.$$

The coefficients c_k in (22) can be stably computed by using the recurrence relation

$$c_{k+1} - 2xc_k + c_{k-1} = 2b_k, \quad k = N-1, N-2, \ldots, 1, \tag{23}$$

in the backward direction with the starting values $c_{N-1} = c_N = 0$. We have omitted the dependence of c_k on N and x. From (7), (8), (9), (18) and (20) we have integration formulas $J_N^{(i)}(f;x) = J^{(i)}(p_N;x)$ to $J^{(i)}(f;x)$ $(i = 2, 3, 4)$ as follows

$$\begin{aligned} J_N^{(2)}(f;x) &= \pi(1-x^2)c_0/2 - \pi x b_0 - \pi a_0^N/2, \\ J_N^{(3)}(f;x) &= \pi(1+x)c_0/2 + \pi b_0/2, \\ J_N^{(4)}(f;x) &= \pi(1-x)c_0/2 - \pi b_0/2. \end{aligned}$$

2.4. Computational complexity

Here we consider the computational complexities (the number of floating-point arithmetic operations (FL)) required to evaluate the approximations to $I^{(i)}(f;x)$ and $J^{(i)}(f;x)$ for each value of x and $1 \le i \le 4$. For a fixed value of N the evaluation of a_k^N, $0 \le k \le N$, (14) requires $O(N \log N)$ FLs by using the FFT. The computation of b_k by the recurrence relations (19) requires $O(N)$ FLs for each value of x. Similarly $O(N)$ FLs are required for the computation of c_k by (23).

3. Error analysis

3.1. Interpolation error

Let $\mathcal{E}_\rho$ denote an ellipse in the complex plane

$$\mathcal{E}_\rho : z = (u + u^{-1})/2, \quad u := \rho e^{i\xi}, \quad 0 \le \xi \le 2\pi, \tag{24}$$

whose foci are at $z = \pm 1$ and the sum of semi-axes is $\rho > 1$. Assume that $f(z)$ is single-valued and analytic inside and on $\mathcal{E}_\rho$. Then, the error of the interpolating polynomial $p_N(t)$ (12) can be expressed in terms of a contour integral [5, 6], which is also expanded in a Chebyshev series [12]:

$$f(t) - p_N(t) = \frac{1}{2\pi i} \oint_{\mathcal{E}_\rho} \frac{W_{N+1}(t)\, f(z)\, dz}{W_{N+1}(z)\,(z-t)} = W_{N+1}(t) \sum_{k=0}^{\infty}{}' \, V_k^N(f)\, T_k(t), \tag{25}$$

where the coefficients $V_k^N(f)$ are given by

$$V_k^N(f) = \frac{1}{\pi^2 i} \oint_{\mathcal{E}_\rho} \frac{\widetilde{U}_k(z)\, f(z)\, dz}{W_{N+1}(z)}, \quad k \ge 0. \tag{26}$$

The Chebyshev function of the second kind, $\widetilde{U}_k(z)$, is defined by

$$\widetilde{U}_k(z) = \int_{-1}^{1} \frac{T_k(x)\, dx}{(z-x)\sqrt{1-x^2}} = \frac{2\pi}{(u-u^{-1})u^k}.$$

3.2. Errors of the approximations to Cauchy principal value integrals

Using (25) in (2) with f being replaced by $f - p_N$ yields the error of the approximation $I^{(1)}(p_N;x)$ (18):

$$I^{(1)}(f;x) - I^{(1)}(p_N;x) = I^{(1)}(f - p_N;x) = \sum_{k=0}^{\infty}{}' \, \Omega_k^N(x)\, V_k^N(f), \tag{27}$$

where $\Omega_k^N(x)$ is given by

$$\Omega_k^N(x) = ⨍_{-1}^{1} \frac{w_1(t) W_{N+1}(t)\, T_k(t)\, dt}{t-x}, \quad -1 < x < 1. \tag{28}$$

Now, we show that $|\Omega_k^N(x)|$ is bounded independently of x.

Lemma 3.1. *Let $\Omega_k^N(x)$ be defined by (28). Then $\Omega_k^N(x)$ is bounded independently of the value of x as well as N and k as follows:*

$$|\Omega_k^N(x)| \leq 2\pi. \tag{29}$$

Proof. From (13) and

$$2T_n(t)T_m(t) = T_{n+m}(t) + T_{|n-m|}(t), \tag{30}$$

we have

$$2W_{N+1}(t)T_k(t) = W_{N+k+1}(t) \bigoplus W_{|N-k|+1}(t), \tag{31}$$

where

$$a(N,k) \bigoplus b(N,k) := \begin{cases} a(N,k) + b(N,k), & N > k, \\ a(N,k) - b(N,k), & N < k, \\ a(N,k), & N = k. \end{cases}$$

From (10) and (31) we see that

$$\begin{aligned} 2\Omega_k^N(x) &= 2\int_{-1}^{1} \frac{w_1(t)\{W_{N+1}(t)\, T_k(t) - W_{N+1}(x)\, T_k(x)\}}{t-x}\, dt \\ &= \int_{-1}^{1} \frac{w_1(t)\{\{W_{N+k+1}(t) - W_{N+k+1}(x)\}}{t-x}\, dt \\ &\bigoplus \int_{-1}^{1} \frac{w_1(t)\{\{W_{|N-k|+1}(t) - W_{|N-k|+1}(x)\}}{t-x}\, dt. \end{aligned} \tag{32}$$

Elliott [5] gives an identity involving the Chebyshev polynomial of the second kind $U_k(t) = \sin(k+1)\theta/\sin\theta$, where $t = \cos\theta$, as follows

$$T_{k+1}(t) - T_{k+1}(x) = 2(t-x)\sum_{j=0}^{k}{}' \, U_{k-j}(x)T_j(t). \tag{33}$$

Using (33) and the relation $U_k(t) - U_{k-2}(t) = 2T_k(t)$ $(k \geq 1)$, where we define $U_{-1}(t) = 0$, we have

$$\frac{W_{k+1}(t) - W_{k+1}(x)}{t-x} = 4\sum_{j=0}^{k}{}'' \, T_{k-j}(x)T_j(t). \tag{34}$$

By noting that $|T_k(x)| \leq 1$ we can verify (29) since using (34) in (32) yields

$$\Omega_k^N(x) = \pi\{T_{N+k}(x) \bigoplus T_{|N-k|}(x)\}. \quad \square \tag{35}$$

Theorem 3.2. *Suppose that $f(z)$ is single-valued and analytic inside and on $\mathcal{E}_\rho$ defined by (24) and let $K = \max_{z\in\mathcal{E}_\rho}|f(z)|$. Then the approximation $I^{(1)}(p_N;x)$ given by (15) uniformly converges to $I^{(1)}(f;x)$ given by (2) as $N \to \infty$,*

$$|I^{(1)}(f;x) - I^{(1)}(p_N;x)| \leq \frac{4\pi K\rho}{(\rho-1)^2(\rho^N - \rho^{-N})} = O(\rho^{-N}), \quad \rho > 1. \tag{36}$$

PROOF. Since $T_k(z) = (u^k + u^{-k})/2$ we have

$$W_{N+1}(z) = T_{N+1}(z) - T_{N-1}(z) = (u^N - u^{-N})(u - u^{-1})/2.$$

Since by noting that $dz/du = (1 - u^{-2})/2$ we have $V_k^N(f)$ (26) written by

$$V_k^N(f) = \frac{2}{\pi i} \oint_{|u|=\rho} \frac{f(z)\, du}{(u - u^{-1})(u^N - u^{-N})u^{k+1}},$$

it follows that

$$\begin{aligned} |V_k^N(f)| &\leq \frac{2}{\pi} \oint_{|u|=\rho} \frac{|f(z)|\,|dz|}{(|u| - |u|^{-1})(|u|^N - |u|^{-N})|u|^{k+1}} \\ &\leq \frac{4K}{(\rho - \rho^{-1})(\rho^N - \rho^{-N})\rho^k}. \end{aligned} \tag{37}$$

Using (29) and (37) in (27) we can easily verify (36). □

Lemma 3.3. *Under the assumption in Theorem 3.2 we have*

$$\left| \int_{-1}^{1} w_1(t)\{f(t) - p_N(t)\}dt \right| \leq \frac{\pi}{2}\{|V_{N+1}^N(f)| + |V_{N-1}^N(f)|\} = O(\rho^{-2N}), \tag{38}$$

$$\left| \int_{-1}^{1} w_1(t)t\{f(t) - p_N(t)\}dt \right| \leq \frac{\pi}{4}\{|V_{N+2}^N(f)| + |V_{N-2}^N(f)|\} = O(\rho^{-2N}). \tag{39}$$

PROOF. From (25) we have

$$\int_{-1}^{1} w_1(t)\{f(t) - p_N(t)\}dt = \sum_{k=0}^{\infty}{}' V_K^N(f) \int_{-1}^{1} w_1(t)\, W_{N+1}(t)\, T_k(t)dt. \tag{40}$$

Since using (13) and (31) in (40) and noting (11) we have

$$\int_{-1}^{1} w_1(t)\, W_{N+1}(t)\, T_k(t)dt = \frac{\pi}{2}(\delta_{k,N+1} - \delta_{k,N-1}),$$

we have (38) by using (37). Similarly we can verify (39) by noting (30) and $t = T_1(t)$. □

From Theorem 3.2, Lemma 3.3 and (4) it follows that $|I^{(2)}(f;x) - I^{(2)}(p_N;x)| = O(\rho^{-N})$. Similarly $|I^{(i)}(f;x) - I^{(i)}(p_N;x)| = O(\rho^{-N})$ $(i = 3, 4)$.

3.3. Errors of the approximations to Hadamard finite-part integrals

Theorem 3.4. *Under the assumption in the Theorem 3.2 the approximation $J^{(1)}(p_N;x)$ (21) uniformly converges to $J^{(1)}(f;x)$ (6) as $N \to \infty$*

$$\begin{aligned} |J^{(1)}(f;x) - J^{(1)}(p_N;x)| &\leq \frac{4\pi K\rho}{(\rho - 1)^2(\rho^N - \rho^{-N})}\left\{N^2 + \frac{2\rho}{(\rho - 1)^2}\right\} \\ &= O(N^2\rho^{-N}), \quad \rho > 1. \end{aligned} \tag{41}$$

Proof. From (6) and (27) we have

$$J^{(1)}(f;x) - J^{(1)}(p_N;x) = J^{(1)}(f - p_N;x) = \sum_{k=0}^{\infty}{}' \frac{d}{dx}\Omega_k^N(x)\, V_k^N(f). \tag{42}$$

Since from (35) we have

$$\begin{aligned}\left|\frac{d\Omega_k^N(x)}{dx}\right| &\le \pi\{(N+k)|U_{N+k-1}(x)| + |N-k|\,|U_{|N-k|-1}(x)|\} \\ &\le \pi\{(N+k)^2 + (N-k)^2\} = 2\pi(N^2+k^2),\end{aligned}$$

we can establish (41) from (37) and (42). □

From Theorem 3.2, Lemma 3.3, Theorem 3.4, (7), (8) and (9) it follows that $|J^{(i)}(f;x) - J^{(i)}(p_N;x)| = O(N^2\rho^{-N})$ $(i = 2, 3, 4)$.

3.4. Errors of approximations to hyper-singular integrals

Here we discuss the errors of approximations $Q^{(n,i)}(p_N;x)$ to $Q^{(n,i)}(f;x)$ $(n \ge 2)$ defined by (1), particularly for the case where $i = 1$. From (1), (27) and (35) we have the error of $Q^{(n,1)}(p_N;x)$

$$\begin{aligned} Q^{(n,1)}(f;x) - Q^{(n,1)}(p_N;x) &= Q^{(n,1)}(f - p_N;x) \\ &= \sum_{k=0}^{\infty}{}' \frac{d^n}{dx^n}\Omega_k^N(x)\, V_k^N(f) \\ &= \pi \sum_{k=0}^{\infty}{}' \frac{d^n}{dx^n}\{T_{N+k}(x) \oplus T_{|N-k|}(x)\}\, V_k^N(f). \end{aligned} \tag{43}$$

Lemma 3.5. *For the Chebyshev polynomial $T_k(x)$ we have the bound of the n-th derivative of $T_k(x)$ as follows*

$$\left|\frac{d^n}{dx^n}T_k(x)\right| < \frac{k^{2n}}{(2n-1)!!}, \quad -1 \le x \le 1, \quad k \ge n, \tag{44}$$

where $(2n-1)!! = (2n-1)(2n-3)\cdots 3\cdot 1$.

Proof. The Chebyshev polynomial $T_k(x)$ can be defined in terms of the Jacobi polynomial $P_k^{(\alpha,\beta)}(t)$ of degree k [1, p.778] as follows

$$T_k(x) = \begin{pmatrix} k-1/2 \\ k \end{pmatrix}^{-1} P_k^{(-1/2,-1/2)}(t) = \frac{k!\,\Gamma(1/2)}{\Gamma(k+1/2)} P_k^{(-1/2,-1/2)}(t). \tag{45}$$

From (45) and

$$\frac{d}{dx}P_k^{(\alpha,\alpha)}(x) = \frac{k+2\alpha+1}{2}P_{k-1}^{(\alpha+1,\alpha+1)}(x), \tag{46}$$

see Szegö [18, p.63], we have

$$\frac{d^n}{dx^n}T_k(x) = \frac{(k+n-1)!\,k!\,\Gamma(1/2)}{2^n\,(k-1)!\,\Gamma(k+1/2)}P_{k-n}^{(n-1/2,n-1/2)}(x). \tag{47}$$

Since

$$\max_{-1\le x\le 1}|P_k^{(\alpha,\alpha)}(x)| = \left(\begin{array}{c} k+\alpha \\ k \end{array}\right) = \frac{\Gamma(k+\alpha+1)}{k!\,\Gamma(\alpha+1)}, \quad \alpha \ge -1/2,$$

see Szegö [18, p.168], it follows from (47) that

$$\begin{aligned}\left|\frac{d^n}{dx^n}T_k(x)\right| &\le \frac{(k+n-1)!}{2^n\,(k-1)!}\cdot\frac{k!\,\Gamma(1/2)}{(k-n)!\,\Gamma(n+1/2)} \\ &= \frac{1}{(2n-1)!!}\prod_{i=0}^{n-1}(k^2-i^2) < \frac{k^{2n}}{(2n-1)!!}. \quad \square\end{aligned}$$

From (43) and (44) we have the following theorem.

Theorem 3.6. *Let $Q^{(n,1)}(p_N;x)$ be an approximation to $Q^{(n,1)}(f;x)$ (1). Then under the assumption in Theorem 3.2 we have an asymptotic error estimate for $Q^{(n,1)}(p_N;x)$ as follows*

$$|Q^{(n,1)}(f;x) - Q^{(n,1)}(p_N;x)| = O(N^{2n}\rho^{-N}) \to 0 \quad (N\to\infty), \quad \rho > 1.$$

4. Numerical Examples

Table 1: Errors of the principal value integrals $I^{(1)}(f_1;x)$

a	x	Integral	N	Error
0.7	0.45	5. 11422 05988 67105	65	1.3×10^{-9}
	0.65	7. 58315 46810 78811	65	-1.2×10^{-9}
	0.85	14. 66076 57167 5237	65	-3.5×10^{-9}
	0.99	42. 29067 03367 8566	65	-7.3×10^{-9}
0.85	0.45	5. 57776 24136 84228	129	-3.2×10^{-9}
	0.65	8. 64891 90463 20080	129	-1.6×10^{-8}
	0.85	19. 24579 28328 0233	129	2.4×10^{-9}
	0.99	135. 20778 50912 062	129	1.6×10^{-7}

The examples in this section are computed in double precision: the machine precision is $2^{-52} = 2.22\ldots\times10^{-16}$. Two test functions $f_1(t)$ and $f_2(t)$ below are used,

$$f_1(t) = \frac{1-a^2}{1-2at+a^2}, \quad f_2(t) = \frac{1}{a^2+t^2}.$$

Table 2: Errors of the principal value integrals $I^{(2)}(f_1;x)$

a	x	Integral	N	Error
0.7	0.45	0. 46575 93759 68254	65	1.1×10^{-9}
	0.65	0. 13812 17459 76793	65	-6.8×10^{-10}
	0.85	−0. 80110 61266 65397	65	-9.7×10^{-10}
	0.99	−4. 46770 72448 64715	65	-1.5×10^{-10}
0.85	0.45	0. 36419 50752 46441	129	-2.6×10^{-9}
	0.65	0. 28236 17688 65156	129	-9.2×10^{-9}
	0.85	0. 00000 00000 00000	129	6.7×10^{-10}
	0.99	−3. 08989 55592 90213	129	3.1×10^{-9}

Table 3: Errors of the principal value integrals $I^{(3)}(f_1;x)$

a	x	Integral	N	Error
0.7	0.45	10. 55721 25219 4710	65	1.9×10^{-9}
	0.65	15. 65379 78773 6983	65	-1.9×10^{-9}
	0.85	30. 26400 92295 8167	65	-6.5×10^{-9}
	0.99	87. 30002 66237 9326	65	-1.5×10^{-8}
0.85	0.45	11. 22934 81534 3192	129	-4.7×10^{-9}
	0.65	17. 41230 90800 1793	129	-2.6×10^{-8}
	0.85	38. 74630 93942 7411	129	4.5×10^{-9}
	0.99	272. 20508 49850 902	129	3.1×10^{-7}

4.1. Numerical example 1

For $f_1(t) = (1-a^2)/(1-2at+a^2)$, where $a = 0.7$ and 0.85, we compute $I^{(i)}(f_1;x)$ and $J^{(1)}(f_1;x)$, namely

$$I^{(i)}(f_1;x) = ⨍_{-1}^{1} \frac{1-a^2}{1-2at+a^2} \cdot \frac{w_i(t)}{t-x} dt, \quad 1 \le i \le 4, \tag{48}$$

$$J^{(1)}(f_1;x) = ⨎_{-1}^{1} \frac{1-a^2}{1-2at+a^2} \cdot \frac{w_1(t)}{(t-x)^2} dt. \tag{49}$$

Each integral above is calculated as

$$I^{(1)}(f_1;x) = \frac{2\pi a}{1-2ax+a^2}, \qquad I^{(2)}(f_1;x) = \frac{\pi(a-x)(1-a^2)}{1-2ax+a^2},$$

$$I^{(3)}(f_1;x) = \frac{\pi(1+a)^2}{1-2ax+a^2}, \qquad I^{(4)}(f_1;x) = \frac{-\pi(1-a)^2}{1-2ax+a^2},$$

$$J^{(1)}(f_1;x) = \frac{4\pi a^2}{(1-2ax+a^2)^2}.$$

In Tables 1∼5, The values of the third column are exact values of the integrals. The fourth and fifth columns give the numbers of function evaluations and the errors of computed results, respectively.

Table 4: Errors of the principal value integrals $I^{(4)}(f_1; x)$

a	x	Integral	N	Error
0.7	0.45	$-0.\ 32877\ 13242\ 12885$	65	7.4×10^{-10}
	0.65	$-0.\ 48748\ 85152\ 12209$	65	-4.1×10^{-10}
	0.85	$-0.\ 94247\ 77960\ 76938$	65	-5.3×10^{-10}
	0.99	$-2.\ 71868\ 59502\ 21936$	65	-7.3×10^{-11}
0.85	0.45	-0.07382332606346774	129	-1.8×10^{-9}
	0.65	-0.1144709873777658	129	-5.5×10^{-9}
	0.85	-0.2547237286694428	129	3.6×10^{-10}
	0.99	-1.789514802677730	129	1.6×10^{-9}

Table 5: Errors of the finite-part integrals $J^{(1)}(f_1; x)$

a	x	Integral	N	Error
0.7	0.45	8. 32547 53935 04590	65	2.0×10^{-8}
	0.65	18. 30416 64715 6954	65	-1.3×10^{-7}
	0.85	68. 41690 66781 7770	65	-1.9×10^{-7}
	0.99	569. 29748 53028 836	65	4.3×10^{-6}
0.85	0.45	9. 90307 68702 48761	129	1.4×10^{-6}
	0.65	23. 81078 92773 1844	129	5.7×10^{-7}
	0.85	117. 90215 42910 413	129	8.8×10^{-6}
	0.99	5819. 06923 17734 28	129	1.7×10^{-4}

4.2. Numerical example 2

Table 6: Errors of the principal value integrals $I^{(1)}(f_2; x)$

a	x	Integral	N	Error
0.5	0.45	$-5.\ 58880\ 28799\ 44003$	49	1.0×10^{-9}
	0.65	$-5.\ 43182\ 70039\ 27402$	49	4.0×10^{-10}
	0.99	$-4.\ 52292\ 76213\ 18798$	49	-1.7×10^{-9}
0.25	0.45	$-20.\ 70198\ 71415\ 2927$	97	1.4×10^{-10}
	0.65	$-16.\ 33868\ 17188\ 0145$	97	8.7×10^{-10}
	0.99	$-11.\ 57611\ 59634\ 6784$	97	-4.5×10^{-9}

For $f_2(t) = 1/(t^2 + a^2)$, where $a = 0.5$ and 0.25, we compute $I^{(i)}(f_2; x)$ and $J^{(1)}(f_2; x)$, namely

$$I^{(i)}(f_2; x) = ⨍_{-1}^{1} \frac{1}{t^2 + a^2} \cdot \frac{w_i(t)}{t - x} dt, \quad 1 \le i \le 4, \tag{50}$$

$$J^{(1)}(f_2; x) = ⨎_{-1}^{1} \frac{1}{t^2 + a^2} \cdot \frac{w_1(t)}{(t - x)^2} dt. \tag{51}$$

Table 7: Errors of the principal value integrals $I^{(2)}(f_2;x)$

a	x	Integral	N	Error
0.5	0.45	$-6.\ 98600\ 35999\ 30005$	49	8.2×10^{-10}
	0.65	$-6.\ 78978\ 37549\ 09252$	49	2.3×10^{-10}
	0.99	$-5.\ 65365\ 95266\ 48499$	49	-3.2×10^{-11}
0.25	0.45	$-21.\ 99586\ 13378\ 7485$	97	1.1×10^{-10}
	0.65	$-17.\ 35984\ 93262\ 2654$	97	5.0×10^{-10}
	0.99	$-12.\ 29962\ 32111\ 8457$	97	-8.9×10^{-11}

Table 8: Errors of the principal value integrals $I^{(3)}(f_2;x)$

a	x	Integral	N	Error
0.5	0.45	$-2.\ 48391\ 23910\ 86224$	49	1.5×10^{-9}
	0.65	$-3.\ 34266\ 27716\ 47632$	49	6.7×10^{-10}
	0.99	$-3.\ 38077\ 41815\ 91829$	49	-3.3×10^{-9}
0.25	0.45	$-17.\ 82671\ 11496\ 5020$	97	2.0×10^{-10}
	0.65	$-14.\ 76765\ 46304\ 5516$	97	1.4×10^{-9}
	0.99	$-10.\ 84530\ 05617\ 3375$	97	-9.0×10^{-9}

Each integral above is calculated as

$$I^{(1)}(f_2;x) = \frac{-\pi x}{a\sqrt{1+a^2}} f_2(x), \qquad I^{(2)}(f_2;x) = \frac{-\pi x\sqrt{1+a^2}}{a} f_2(x),$$

$$I^{(3)}(f_2;x) = \frac{\pi(a^2-x)}{a\sqrt{1+a^2}} f_2(x), \qquad I^{(4)}(f_2;x) = \frac{-\pi(a^2+x)}{a\sqrt{1+a^2}} f_2(x),$$

$$J^{(1)}(f_2;x) = \frac{\pi(x^2-a^2)}{a\sqrt{1+a^2}} \{f_2(x)\}^2.$$

Tables 6∼10 show the computed results of $I^{(i)}(f_2;x)$ $(i = 1, \ldots, 4)$ and $J^{(1)}(f_2;x)$,

Table 9: Errors of the principal value integrals $I^{(4)}(f_2;x)$

a	x	Integral	N	Error
0.5	0.45	$-8.\ 69369\ 33688\ 01783$	49	5.7×10^{-10}
	0.65	$-7.\ 52099\ 12362\ 07172$	49	1.4×10^{-10}
	0.99	$-5.\ 66508\ 10610\ 45768$	49	-1.6×10^{-11}
0.25	0.45	$-23.\ 57726\ 31334\ 0834$	97	7.8×10^{-11}
	0.65	$-17.\ 90970\ 88071\ 4774$	97	3.0×10^{-10}
	0.99	$-12.\ 30693\ 13652\ 0192$	97	-4.5×10^{-11}

respectively.

Table 10: Errors of the principal value integrals $J^{(1)}(f_2; x)$

a	x	Integral	N	Error
0.5	0.45	−1. 30371 09234 98294	49	8.4×10^{-8}
	0.65	2. 14352 91007 77756	49	-5.8×10^{-8}
	0.99	2. 71160 46706 59817	49	2.3×10^{-6}
0.25	0.45	24. 30421 97049 4003	97	-1.3×10^{-7}
	0.65	18. 65797 11935 5598	97	1.2×10^{-7}
	0.99	10. 29113 69672 9242	97	3.2×10^{-6}

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