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Abstract

For the finite Hilbert transform of oscillatory functions $Q(f;c,\omega) = ⨍_{-1}^{1} f(x) e^{i\omega x}/(x-c)\,dt$ with a smooth function f and real $\omega \neq 0$, for $c \in (-1,1)$ in the sense of Cauchy principal value or for $c = \pm 1$ of Hadamard finite-part, we present an approximation method of Clenshaw-Curtis type and its algorithm. Interpolating f by a polynomial p_n of degree n and expanding in terms of the Chebyshev polynomials with $O(n \log n)$ operations by the FFT, we obtain an approximation $Q(p_n;c,\omega) \cong Q(f;c,\omega)$. We write $Q(p_n;c,\omega)$ as a sum of the sine and cosine integrals and an oscillatory integral of a polynomial of degree $n-1$. We efficiently evaluate the oscillatory integral with a combination of authors' previous method and Keller's method. For $f(z)$ analytic on the interval $[-1,1]$ in the complex plane z, the error of $Q(p_n;c,\omega)$ is bounded uniformly with respect to c and ω. Numerical examples illustrate the performance of our method.

Keywords: quadrature rule, principal value integral, oscillatory function, Chebyshev interpolation, error analysis, uniform approximation,
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1. Introduction

Integrals $\int_{-1}^{1} f(t)K(t)\,dt$ of a real-valued, regular function $f(t)$ and the singular or/and oscillatory kernel $K(t) = 1/(t-c)$, $e^{i\omega t}$ ($i = \sqrt{-1}$, real $\omega \neq 0$) or

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$e^{i\omega t}/(t-c)$, often arise in scientific computing (cf. [1, 1.6, 2.10]).

In this work, we consider integrals of $K(t) = e^{i\omega t}/(t-c)$ ($c \in [-1,1]$), finite Hilbert transform of highly-oscillatory functions (assuming $|\omega| \gg 1$)

$$Q(f;c,\omega) := \begin{cases} \displaystyle ⨍_{-1}^{1} \frac{f(t)\, e^{i\omega t}}{t-c}\, dt, & c \in (-1,1), \\ \displaystyle ⨎_{-1}^{1} \frac{f(t)\, e^{i\omega t}}{t-c}\, dt, & c = \pm 1, \end{cases} \tag{1.1}$$

in the sense of Cauchy principal value (CPV) (cf. [1, p.11]) for $c \in (-1,1)$, or of Hadamard finite-part (HFP) (cf. [1, 1.6.1]) for $c = \pm 1$. High oscillation and singularity make it difficult to approximate the integrals with well-known quadrature rules such as the Gaussian and Newton-Cotes rules.

For efficiently approximating $Q(f;c,\omega)$ (1.1), we develop a method, an extended Clenshaw–Curtis (C–C) quadrature rule [2] (cf. [3, Sec.8.3]). We use approximation methods [4–6] for CPV integrals and an algorithm which is a combination of two methods given in [7, 8] for evaluating oscillatory integrals (cf. [9, 10]). For a C–C type method for CPV integrals with logarithmic singularity, see [11]. Since the C–C rule has some advantages shown below, several C–C type methods for $Q(f;c,\omega)$ have been developed (cf. [12–14]). We review them later. For some works other than the C–C type method, see [15–17].

The C–C rule is an interpolatory integration rule. For an integer $n \geq 1$, let p_n be a polynomial interpolating f at the Chebyshev points $x_{n,j} = \cos(\pi j/n)$ ($0 \leq j \leq n$) on $[-1,1]$. Then, we obtain an approximation $Q(p_n;c,\omega) \cong Q(f;c,\omega)$ in (1.1). We expand p_n in terms of the Chebyshev polynomials of the first kind $T_k(t) = \cos k\theta$ ($t = \cos\theta$),

$$f(t) \cong p_n(t) = \sum_{k=0}^{n}{}'' a_k T_k(t), \tag{1.2}$$

where the double prime indicates the summation whose first and last terms are halved. The coefficients a_k are efficiently evaluated with $O(n \log_2 n)$ arithmetic operations with the fast Fourier transform (FFT) (cf. [18],[19, p.232]). If $f(t)$ is a sufficiently smooth function, then p_n in (1.2) converges rapidly to f as n grows. We can effectively increase the accuracy of the approximation by doubling the number of the nodes $\{x_{n,j}\}$ $(0 \le j \le n)$ for p_n. Indeed, these nodes are reused in $\{x_{2n,j}\}$ $(0 \le j \le 2n)$ for p_{2n}, since $\{x_{2n,j}\}_{j=0}^{2n} = \{x_{n,j}\}_{j=0}^{n} \cup \{\cos[\pi(2j+1)/(2n)]\}_{j=0}^{n-1}$; see [20–22], [23, p.75].

To evaluate the approximation $Q(p_n; c, \omega)$, every method of C–C type in [12–14] makes use of a *recurrence relation* (RR), although a numerical instability often occurs in the computation of the RR (cf. [24, Section 6]). We review them and ours.

Keller [13] approximates the indefinite integral,

$$\int \frac{(p_n(t) - R)e^{i\omega t}}{t - c}\, dt \cong e^{i\omega t} \sum_{k=0}^{N}{}' \alpha_k T_k(t) \quad (-1 \le t \le 1), \tag{1.3}$$

where $N = n(1 + o(1))$ and $R \cong p_n(c)$. The prime indicates the summation whose first term is halved. The coefficients α_k satisfy the five-term RR,

$$\begin{aligned} \frac{i\omega}{2}\alpha_{k-2} + (k - 1 - i\omega c)\, \alpha_{k-1} - 2kc\, \alpha_k + (k + 1 + i\omega c)\, \alpha_{k+1} - \frac{i\omega}{2}\alpha_{k+2} \\ = a_{k-1} - a_{k+1} \quad (k \ge 2). \end{aligned} \tag{1.4}$$

Then, $Q(p_n; c, \omega)$ for $c \in (-1, 1)$ is given by

$$Q(p_n; c, \omega) \cong e^{i\omega} \sum_{k=0}^{N}{}' \alpha_k - e^{-i\omega} \sum_{k=0}^{N}{}' (-1)^k \alpha_k + R \fint_{-1}^{1} \frac{e^{i\omega t}}{t - c}\, dt, \tag{1.5}$$

where $R = (a_0 - a_2)/2 - i\omega c(\alpha_2 - \alpha_0)/2 - i\omega(\alpha_1 - \alpha_3)/4 - \alpha_2 + c\alpha_1$. The integrals on the right of (1.5) is simple to evaluate, since they are expressed by the sine and cosine integrals or the exponential integral; see Section 2 and [12–14].

Other works in [12, 14] similarly and exactly divide $Q(p_n; c, \omega)$ into a non-singular integral $I(p_n; c, \omega)$ and CPV or HFP integrals,

$$Q(p_n; c, \omega) = I(p_n; c, \omega) + p_n(c) \begin{cases} \displaystyle \fint_{-1}^{1} \frac{e^{i\omega t}}{t - c}\, dt & (c \in (-1, 1)), \\ \displaystyle \ddashint_{-1}^{1} \frac{e^{i\omega t}}{t - c}\, dt & (c = \pm 1), \end{cases} \tag{1.6}$$

$$I(p_n; c, \omega) := \int_{-1}^{1} \frac{p_n(t) - p_n(c)}{t - c} e^{i\omega t}\, dt. \tag{1.7}$$

Clenshaw's algorithm effectively evaluate $p_n(c)$ at $t = c$ (cf. [3, p.27]); see (5.1). The problem is to efficiently evaluate the integral $I(p_n; c, \omega)$ (1.7).

Wang, Zhang and Huybrechs [14], an improvement of the method by Wang and Xiang [12], make use of the key relation (cf. [4])

$$\frac{p_n(t) - p_n(c)}{t - c} = {\sum_{k=0}^{n-1}}' b_k T_k(t) =: q_{n-1}(t), \tag{1.8}$$

to obtain a simple form of the integral in (1.7),

$$I(p_n; c, \omega) = \int_{-1}^{1} q_{n-1}(t)\, e^{i\omega t}\, dt = {\sum_{k=0}^{n-1}}' b_k \nu_k, \quad \nu_k := \int_{-1}^{1} T_k(t) e^{i\omega t}\, dt. \tag{1.9}$$

The coefficients b_k in (1.8) are obtained by backward computation of the three-term RR (3.2) below. The moments ν_k in (1.9) are given by

$$\nu_k = \frac{1}{i\omega}\{e^{i\omega} - (-1)^k e^{-i\omega}\} - \frac{k}{i\omega}\mu_{k-1} \quad (k \geq 1),$$

where $\mu_k = \int_{-1}^{1} U_k(t) e^{i\omega t}\, dt$, and $U_k(t) = \sin(k+1)\theta / \sin\theta$ ($t = \cos\theta$) is the Chebyshev polynomial of the second kind of degree k (cf. [3, p.3]). The moments μ_k satisfy the three-term RR,

$$\mu_k + \frac{2k}{i\omega}\mu_{k-1} - \mu_{k-2} = \frac{2}{i\omega}\{e^{i\omega} - (-1)^k e^{-i\omega}\} \quad (k \geq 2). \tag{1.10}$$

To evaluate $I(p_n; c, \omega)$ in (1.9), we show our previous method [8] now. Later we give an improved algorithm with a smaller number of arithmetic operations. We expand the indefinite integral in terms of the Chebyshev polynomials,

$$\int q_{n-1}(t) e^{i\omega t}\, dt = \frac{e^{i\omega t}}{i\omega}\varphi(t), \quad \varphi(t) := {\sum_{k=0}^{\infty}}' d_k T_k(t). \tag{1.11}$$

Then, we obtain

$$I(p_n; c, \omega) = \frac{e^{i\omega}\varphi(1) - e^{-i\omega}\varphi(-1)}{i\omega} = \frac{e^{i\omega}}{i\omega}{\sum_{k=0}^{\infty}}' d_k - \frac{e^{-i\omega}}{i\omega}{\sum_{k=0}^{\infty}}' (-1)^k d_k. \tag{1.12}$$

Since $\varphi(t)$ in (1.11) is an entire function (cf. [8]), the coefficients d_k converge very fast to 0 as $k \to \infty$. They satisfy the three-term RR,

$$d_{k-1} + \frac{2k}{i\omega} d_k - d_{k+1} = b_{k-1} - b_{k+1} \qquad (k = 1, 2, \ldots), \tag{1.13}$$

with the coefficients b_k of $q_{n-1}(t)$ in (1.8) and $b_k = 0$ ($k \geq n$); see [8, Theorem 2.1]. Note that the left-hand sides of (1.10) and (1.13) are almost the same.

The numerical instability of the RRs (1.4), (1.10) and (1.13) is a serious problem. See [24, Section 8] and [25, Section 3.3] for the stability of RRs of an arbitrary order. If $n-1 \le |\omega|$, then forward and backward computations of the RRs are numerically stable (cf. [26]). But, when $n-1 > |\omega|$, neither forward nor backward computation of the RRs is stable owing to severe loss of significant figures occurring in the computation process. To avoid this difficulty it is required to transform the RR into a system of linear equations of size $N \ge n-1$, as shown below. It is desirable to determine N as small as possible. Unique solution of the linear system requires some additional boundary conditions (cf. [25, 27–29]).

For the linear system (1.4), Keller [13] uses Lozier's algorithm [25] with one initial condition $\alpha_{\lfloor|\omega|\rfloor} = 0$ and three trailing conditions $\alpha_{N+1} = \alpha_{N+2} = \alpha_{N+3} = 0$ for a properly selected value of N. By assuming the asymptotic convergence rate of $|\alpha_k|$ as $k \to \infty$, the value of N is determined so that

$$\max\{|\alpha_{N-3}|, |\alpha_{N-2}|, |\alpha_{N-1}|, |\alpha_N|\} < \epsilon \max_{-1\le x\le 1} |p_n(x)|/N^2, \tag{1.14}$$

with a machine epsilon ϵ. To estimate the smallest value of N is not easy since it requires to estimate the magnitude of $|\alpha_k|$ accurately and efficiently. The URL to Matlab codes implementing the method is shown in [13].

Wang, Zhang and Huybrechs [14] adopt the method due to Domínguez et al. [29] to solve the linear system (1.10) for $N \ge n-1$. In addition to the trivial boundary condition $\mu_0 = (2\sin\omega)/\omega$, a sufficiently accurate value of μ_N is needed. Using an asymptotic expansion of μ_N, Domínguez et al. devise an ingenious method to obtain the approximate value $\tilde{\mu}_N$ of μ_N for a sufficiently large N. Again, to determine the smallest value of N is not easy since it requires to estimate the error of $\tilde{\mu}_N$ precisely and efficiently. Moreover, in order to guarantee the accuracy of $\tilde{\mu}_N$, the value of N can not be so small. An algorithm implementing this method is given (cf. [14]).

Finally, we outline our scheme to solve the linear system (1.13). The normalization relation $\varphi(-1) = \sum_{k=0}^{\infty}{}'(-1)^k d_k = 0$ is used in [8] for a boundary condition. Here, we choose the condition $d_{\lfloor|\omega|\rfloor} = 0$ given in [7] that leads to a simpler and faster algorithm. See [7] and Subsection 3.2 and Section 4 for details and a comparison of the computational costs. Since d_k converges rapidly to 0 ($k \to \infty$), instead of the infinite summation (1.11) for $\varphi(t)$, it is sufficient to use a truncated Chebyshev series

$$\varphi^{[N]}(t) = \sum_{k=0}^{N}{}' d_k^{[N]} T_k(t) \approx \varphi(t), \tag{1.15}$$

with an integer $N \geq n-1$. Then, in view of (1.15) we have an approximation $I^{[N]}(p_n; c, \omega)$ to $I(p_n; c, \omega)$ in (1.12),

$$I^{[N]}(p_n; c, \omega) = \frac{e^{i\omega}}{i\omega} \sum_{k=0}^{N}{}' d_k^{[N]} - \frac{e^{-i\omega}}{i\omega} \sum_{k=0}^{N}{}' (-1)^k d_k^{[N]}. \tag{1.16}$$

As shown in Lemma 1.1 (cf. [8, Theorem 2.3]) below, we easily and automatically determine the value of N close to $n-1$ such that the error of $I^{[N]}(p_n; c, \omega)$ is at the level of the unit roundoff u of computer (cf. [30, p.38]).

Lemma 1.1. *The truncation error E_T of the approximation $I^{[N]}(p_n; c, \omega)$ in (1.16), where $N \geq n-1$, is bounded by the last coefficient $d_N^{[N]}$ of $\varphi^{[N]}(x)$ in (1.15) as*

$$E_T := |I(p_n; c, \omega) - I^{[N]}(p_n; c, \omega)| \leq \left|d_N^{[N]}\right|. \tag{1.17}$$

Lemma 1.1 is proved almost in the same way as that in [8]; see Appendix A.

We solve the system of linear equations by Gaussian elimination with $O(N)$ arithmetic operations. We increase N until $\left|d_N^{[N]}\right| \leq$ u is satisfied in the forward elimination process, followed by computing $d_k^{[N]}$ $(0 \leq k \leq N-1)$ by back substitution. The result guarantees that $E_T \leq$ u, from Lemma 1.1.

Lemma 1.1 provides an efficient and verified error bound to determine N. As we have mentioned above, on the other hand, it is not easy to determine the smallest value of N in [13, 14]. In the scheme of Domínguez et al. [29] that is of a different type from ours, the size N of the linear system is fixed, provided a sufficiently accurate approximation $\tilde{\mu}_N$ to μ_N is given. To obtain $\tilde{\mu}_N$, however, the value of N is doubled from $n-1$ until the required accuracy is achieved. Using the Matlab routines by Domínguez et al.,

Table 1: The size N of linear systems for each $n-1$

$n-1=$	80	160	320	640
Domínguez et al.	320	320	640	1280
Ours	**110**	**173**	**322**	**640**

we performed numerical experiments for the integral $\int_{-1}^{1} f(x)e^{i\omega x}\,dx$, where $f(x) = (1-\alpha)/(1-2\alpha x+\alpha^2) = 2\sum_{k=0}^{\infty}{}'\alpha^k T_k(x)$ with $\alpha = 0.9$. We compare the size N of our linear system and N in [29] required to within the unit round-off, $\mathrm{u} \cong 1.11\times 10^{-16}$ of computer. When $\omega = 60$ and $n-1 = 80, 160, 320$ and 640, the required values of N in Domínguez et al. [29] and our method are given in Table 1. The Matlab code by Keller [13], where a strange formula different from (1.14) is used, indicates that $N = 117$ $(n-1 \le 115)$ and $N = n+1$ $(n-1 > 115)$ when $\omega = 60$.

Our method approximates $Q(f;c,\omega)$ uniformly, independently of c and ω; see Theorem 6.5, and is efficiently implemented.

The paper is organized as follows. Section 2 shows that the integrals in (1.6) are expressed by the sine and cosine integrals. Section 3 gives a review of [8] for approximating the oscillatory integrals $I(p_n;c,\omega)$ and its improvement. In Section 4 we rewrite the RR (1.13) in the form of a linear system of finite dimension and solve it by Gaussian elimination via bordering (cf. [31, p.55]). The number of arithmetic operations is examined. Section 5 summarizes the algorithm to compute $Q(p_n;c,\omega)$. In Section 6, assuming that $f(z)$ is analytic on $[-1,1]$ in the complex plane z, we give error analysis of $Q(p_n;c,\omega)$ and a uniform error bound with respect to c and ω. Section 7 gives numerical examples to show the error behavior of $Q(p_n;c,\omega)$.

2. Treatment of CPV and HFP integrals

The CPV and HFP integrals in (1.6) are expressed by the sine and cosine integrals or the exponential integral (cf. [12–16]).

Lemma 2.1. *The CPV integral* $-\!\!\!\!\!\!\int_{-1}^{1} e^{i\omega t}/(t-c)\,dt$ *with* $c \in (-1,1)$ *in (1.6) is written by*

$$-\!\!\!\!\!\!\int_{-1}^{1} \frac{e^{i\omega t}}{t-c}\,dt = e^{ic\omega}\big[\,\mathrm{Ci}(|(1-c)\omega|) - \mathrm{Ci}(|(1+c)\omega|) + i\{\mathrm{Si}((1-c)\omega) + \mathrm{Si}((1+c)\omega)\}\,\big], \quad c \in (-1,1), \tag{2.1}$$

where $\mathrm{Si}(x)$ *and* $\mathrm{Ci}(x)$ *are the sine and cosine integrals (cf. [32, 5.2.1, 5.2.27],[33, p.181,p.187]), respectively, and given by*

$$\mathrm{Si}(x) = \int_0^x \frac{\sin t}{t}\,dt, \qquad \mathrm{Ci}(x) = \begin{cases} -\displaystyle\int_x^{\infty} \frac{\cos t}{t}\,dt & (x > 0), \\ \pi i - \displaystyle -\!\!\!\!\!\!\int_x^{\infty} \frac{\cos t}{t}\,dt & (x < 0). \end{cases} \tag{2.2}$$

See [12, 15] for the proof.

Lemma 2.2. *For the HFP integral* $=\!\!\!\!\!\!\int_{-1}^{1} e^{i\omega t}/(t-c)\,dt$ *with* $c = \pm 1$ *in (1.6), we have*

$$=\!\!\!\!\!\!\int_{-1}^{1} \frac{e^{i\omega t}}{t-c}\,dt = \begin{cases} e^{i\omega c}\big[\,c\,\{\gamma + \log|\omega| - \mathrm{Ci}(2|\omega|)\} + i\,\mathrm{Si}(2\omega)\,\big] & (\omega \neq 0), \\ -c\log 2 & (\omega = 0), \end{cases} \tag{2.3}$$

with Euler's constant $\gamma = 0.5772\cdots$.

The formula similar to (2.3) given in terms of the exponential integral is an immediate consequence of the results presented in [14].

Proof of Lemma 2.2 The case $\omega = 0$ is trivial. For $\omega \neq 0$ we claim that

$$=\!\!\!\!\!\!\int_{-1}^{1} \frac{e^{i\omega(t-c)} - 1}{t-c}\,dt = c\{\gamma + \log(2|\omega|) - \mathrm{Ci}(2|\omega|)\} + i\,\mathrm{Si}(2\omega). \tag{2.4}$$

Then, we verify (2.3) for $\omega \neq 0$ using (2.4) and $=\!\!\!\!\!\!\int_{-1}^{1} 1/(t-c)\,dt = -c\log 2$ in the relation

$$=\!\!\!\!\!\!\int_{-1}^{1} \frac{e^{i\omega t}}{t-c}\,dt = e^{i\omega c}\Big\{\int_{-1}^{1} \frac{e^{i\omega(t-c)} - 1}{t-c}\,dt + =\!\!\!\!\!\!\int_{-1}^{1} \frac{1}{t-c}\,dt\Big\}.$$

It remains to verify (2.4). For $c = \pm 1$, by the change of variables $u = c\omega(c-t) = \omega(1-ct)$ we have

$$\int_{-1}^{1} \frac{e^{i\omega(t-c)}-1}{t-c}\,dt = \int_{(1+c)\omega}^{(1-c)\omega} \frac{e^{-icu}-1}{u}\,du = -c\int_{0}^{2\omega} \frac{e^{-icu}-1}{u}\,du$$
$$= c\int_{0}^{2\omega} \frac{1-\cos u}{u}\,du + i\int_{0}^{2\omega} \frac{\sin u}{u}\,du. \qquad (2.5)$$

Abramowitz & Stegun [32, 5.2.2] gives

$$\int_{0}^{x} \frac{1-\cos u}{u}\,du = \gamma + \log x - \mathrm{Ci}(x) \quad (x > 0).$$

Since the left-hand side above is an even function, it follows that

$$\int_{0}^{x} \frac{1-\cos u}{u}\,du = \gamma + \log|x| - \mathrm{Ci}(|x|) \quad (x \neq 0). \qquad (2.6)$$

Using (2.2) and (2.6) in (2.5) verifies (2.4). □

3. Evaluation of a_k, b_k and $I(p_n; c, \omega)$

We give an expression for the Chebyshev coefficients a_k in (1.2) and b_k in (1.8) followed by the evaluation of $I(p_n; c, \omega)$.

3.1. Evaluation of a_k and b_k

The interpolation condition $f(x_{n,j}) = p_n(x_{n,j})$ $(0 \le j \le n)$ gives the coefficients a_k in (1.2) (cf. [3, p.156]),

$$a_k = \frac{2}{n}\sum_{j=0}^{n}{}'' f(\cos\frac{\pi j}{n})\cos\frac{\pi jk}{n} \quad (0 \le k \le n). \qquad (3.1)$$

The summations on the right are evaluated with $O(n\log_2 n)$ multiplications with the FFT (cf. [18]); see Section 5. If f is sufficiently smooth, then $|a_k|$ decreases rapidly to zero as k grows, therefore p_n in (1.2) converges rapidly to f as n grows (cf. [34, p.175], [21]).

The coefficients b_k for q_{n-1} in (1.8) satisfy the three-term RR (cf. [4]),

$$b_{k-1} - 2c\,b_k + b_{k+1} = 2a_k \quad (k = 1, 2, \ldots, n), \qquad (3.2)$$

with a_n replaced by $a_n/2$. We set $b_k = 0$ $(k \ge n)$ for convenience. Since $c \in [-1, 1]$, backward computation of the RR (3.2) is numerically stable and gives the values of $b_0 \ldots, b_{n-1}$ with starting values $b_n = b_{n+1} = 0$.

3.2. Evaluation of $I(p_n; c, \omega)$

Differentiating both the sides of the first equation in (1.11) gives the linear first-order differential equation for $\varphi(t)$,

$$\varphi'(t) + i\omega\,\varphi(t) = i\omega\, q_{n-1}(t) \quad (-1 \le t \le 1), \tag{3.3}$$

with $q_{n-1}(t)$ in (1.8). Lemma 3.1 gives the general solution of (3.3).

Lemma 3.1 (cf. [8])**.** *The general solution $\varphi(t)$ of the differential equation (3.3) is an entire function given, with a constant C, by*

$$\varphi(t) = Ce^{-i\omega t} + \phi_{n-1}(t), \qquad \phi_{n-1}(t) := \sum_{k=0}^{n-1} \left(\frac{i}{\omega}\right)^k \frac{d^k}{dt^k} q_{n-1}(t). \tag{3.4}$$

We expand the polynomial $\phi_{n-1}(t)$ of degree $n-1$ and $e^{-i\omega t}$ in (3.4) in terms of the Chebyshev polynomials,

$$\phi_{n-1}(t) = {\sum_{k=0}^{n-1}}' d_k^{(0)} T_k(t), \tag{3.5}$$

$$e^{-i\omega t} = 2{\sum_{k=0}^{\infty}}' (-i)^k J_k(\omega)\, T_k(t), \tag{3.6}$$

with the Bessel function of the first kind $J_k(z)$ (cf. [3, p.109], [32, p.358,p.375]). Inserting (3.5) and (3.6) into (3.4) for $\varphi(t)$ and comparing d_k for $\varphi(t)$ in (1.11), we obtain

$$d_k = \begin{cases} 2C\,(-i)^k J_k(\omega) + d_k^{(0)} & (0 \le k \le n-1), \\ 2C\,(-i)^k J_k(\omega) & (n \le k). \end{cases} \tag{3.7}$$

Lemma 3.1 implies two types of the solution $\varphi(t)$ with $C \neq 0$ or $C = 0$. Since the definite integral $I(p_n; c, \omega)$ defined by (1.7) is independent of C, an arbitrary value of C is possible provided $|C| < \infty$. When $C \neq 0$, the function $\varphi(t)$ behaves more like $e^{-i\omega t}$ and the Chebyshev expansion of $\varphi(t)$ in (1.11) converges very fast. This is seen as follows. Since $J_k(z) \sim \{ez/(2k)\}^k/\sqrt{2\pi k}$ $(k \to \infty)$ (cf. [32, p.365]), it follows that $J_k(\omega)$, and, consequently, d_k in (3.7) converge to 0 very fast for $k > |\omega|$ with ω fixed.

Let $M = \lfloor |\omega| \rfloor$. When $n - 1 \le M$, we use the polynomial solution with $C = 0$ in (3.4), that is, $\varphi(t) = \phi_{n-1}(t)$ in (3.5), namely, $d_k = d_k^{(0)}$ $(0 \le k \le$

$n-1$). We set $d_k = 0$ ($k \geq n$). Forward and backward computations of (1.13) are numerically stable for $0 \leq k \leq n-1$. With the conditions $d_n = 0$ and $d_{n-1} = b_{n-1}$, backward recursion of (1.13) gives $d_{n-2}, \cdots, d_0$. Then, we have

$$I(p_n; c, \omega) = \frac{e^{i\omega}}{i\omega} \sum_{k=0}^{n-1}{}' d_k - \frac{e^{-i\omega}}{i\omega} \sum_{k=0}^{n-1}{}' (-1)^k d_k. \tag{3.8}$$

When $n-1 > M$, neither forward nor backward computation of (1.13) is stable for $M < k \leq n-1$. In the computation process the value of d_k is soon contaminated with the rounding error; see [8, 9]. In this case, we need a solution of the second type ($C \neq 0$). We use the infinite Chebyshev series $\varphi(x)$ given by (1.11), where the coefficients d_k satisfy the RR (1.13). To avoid the numerical instability, we reformulate the RR (1.13) as the problem of a system of linear equations. The normalization relation $\varphi(-1) = \sum_{k=0}^{\infty}{}'(-1)^k d_k = 0$ is used in [8].

In this paper, rather than this relation, we prefer the condition $d_M = 0$ (cf. [7]) that leads to reduction of the computational cost; see Section 4. Then, from (3.7) we obtain $C = -d_M^{(0)}/\{2(-i)^M J_M(\omega)\}$. We assure that $|C| < \infty$, since $|d_M^{(0)}| < \infty$ and $J_M(\omega) \neq 0$, see Appendix B.

Since d_k converges rapidly to 0 ($k \to \infty$) as shown above, instead of the infinite summation (1.11) for $\varphi(t)$, we use the truncated Chebyshev series (1.15). We determine $d_k^{[N]}$ ($0 \leq k \leq N$) so that (1.13) is satisfied for $1 \leq k \leq N$ with the condition $d_M^{[N]} = d_M = 0$ and $d_k^{[N]} = 0$ ($k > N$), namely,

$$d_{k-1}^{[N]} + \frac{2k}{i\omega} d_k^{[N]} - d_{k+1}^{[N]} = b_{k-1} - b_{k+1} \quad (1 \leq k \leq N), \tag{3.9}$$

recalling that $b_k = 0$ ($k \geq n$). Then, we have the approximation $I^{[N]}(p_n; c, \omega)$ (1.16). Section 4 shows a system of linear equations converted from (3.9) for $M+1 \leq k \leq N$. We solve the linear system by Gaussian elimination with the smallest N satisfying the condition $\left|d_N^{[N]}\right| \leq \mathrm{u}$, namely, $E_T \leq \mathrm{u}$ from Lemma 1.1. Then, with $d_{M+1}^{[N]}$ obtained above and the condition $d_M^{[N]} = 0$, backward computation of the RR (3.9) for $1 \leq k \leq M$ gives $d_k^{[N]}$ ($0 \leq k \leq M-1$).

4. Computing $d_k^{[N]}$

Recall that $M = \lfloor |\omega| \rfloor$. The linear system converted from the RR (3.9) for $M+1 \le k \le N$ with the condition $d_M^{[N]} = 0$ is solved in the same way as the one given in [7]. So, we omit the details and show only a difference (the stopping criterion) and the computational costs required to compute $d_k^{[N]}$ $(0 \le k \le N)$.

According to the notation in [7], let $\mu_k = 2k/\omega$ and $\alpha_k = b_{k-1} - b_{k+1}$ $(1 \le k \le N)$. Letting $d_M^{[N]} = 0$, we rewrite (3.9) as follows,

$$d_{k-1}^{[N]} - i\mu_k\, d_k^{[N]} - d_{k+1}^{[N]} = \alpha_k \quad (1 \le k \le M), \tag{4.1}$$

$$\left.\begin{aligned} -i\mu_{M+1}\, d_{M+1}^{[N]} - d_{M+2} &= \alpha_{M+1}, \\ d_{k-1}^{[N]} - i\mu_k\, d_k^{[N]} - d_{k+1}^{[N]} &= \alpha_k \quad (M+2 \le k \le N-1), \\ d_{N-1}^{[N]} - i\mu_N\, d_N^{[N]} &= \alpha_N. \end{aligned}\right\} \tag{4.2}$$

Note that μ_k $(1 \le k \le N)$ are stored in a real array.

We solve the tridiagonal linear system (4.2) by Gaussian elimination without pivoting, increasing N until the stopping criterion $\left|d_N^{[N]}\right| \le \mathrm{u}$ is satisfied in the forward elimination process. This means that $E_T \le \mathrm{u}$, from Lemma 1.1.

Keller [7] adopts a different stopping criterion. If we modify his stopping rule in Algorithm 2.4 in [7] to $|\widetilde{\alpha}_{j+1}/\widetilde{\mu}_{j+1}| \le \mathrm{u}$ for some $j \ge M$ ($\widetilde{\alpha}_{j+1}$ and $\widetilde{\mu}_{j+1}$ defined in [7]) and set $N = j+1$, then we obtain practically the same algorithm as ours [1]. The value of $d_N^{[N]}$, that is, $-\widetilde{\alpha}_N/(i\widetilde{\mu}_N)$, is obtained in the forward elimination (2.11) in [7], so requires no additional computation.

Finally, we compute the RR (4.1) in the backward direction to obtain $d_k^{[N]}$ $(0 \le k \le M-1)$.

This process in [7, Lemma 2.2 and Lemma 2.3] to solve (4.2) followed by (4.1) requires a small number of arithmetic operations, namely, $2N - 2M - 1$ complex/reals, $M - 1$ complex×reals and $N - M - 1$ real reciprocals.

Our previous scheme in [8] requires $11N - 8M - 3$ complex multiplication-divisions.

[1]The authors thank the reviewer for pointing this out.

5. Algorithm

We give the algorithm to compute $Q(p_n; c, \omega)$ in (1.6). We assume the double precision computation with the unit roundoff $\mathrm{u} \cong 1.11 \times 10^{-16}$. We assume that standard routines for the FFT, $\mathrm{Ci}(x)$ and $\mathrm{Si}(x)$ are available.

Purpose

Given the function f, the values of $c \in [-1, 1]$ and ω and the number of function evaluations $n + 1 \geq 2$ required, we compute an approximation $Q(p_n; c, \omega)$.

Outline of the algorithm

Output is the approximation $Q(p_n; c, \omega)$.

Step 1. Initialization. Set $\mathrm{u} = 2^{-53} \cong 1.11 \times 10^{-16}$.

Step 2. Use the FFT† to compute the summations on the right-hand side of (3.1) to obtain the Chebyshev coefficients a_k $(0 \leq k \leq n)$ of p_n.

Step 3. Compute the RR (3.2) in the backward direction with starting values $b_n = 0$ and $b_{n-1} = a_n/2$ to obtain b_k $(0 \leq k \leq n-1)$.

Step 4. If $n - 1 \leq |\omega|$, go to Step 5. Otherwise, go to Step 6.

Step 5. Compute the RR (1.13) in the backward direction with starting values $d_n = d_{n+1} = 0$ to obtain d_k $(0 \leq k \leq n-1)$.
Compute the right-hand side of (3.8) to obtain $I(p_n; c, \omega)$.
Go to Step 8.

Step 6. Solve the linear system (4.2) for $d_k^{[N]}$ by Gaussian elimination via bordering with the smallest N that satisfies $|d_N^{[N]}| \leq \mathrm{u}$. Set $d_M^{[N]} = 0$. Compute the RR (4.1) in the backward direction to obtain $d_k^{[N]}$ $(0 \leq k \leq M-1)$.

Step 7. Compute the approximation $I^{[N]}(p_n; c, \omega)$ in (1.16) to $I(p_n; c, \omega)$.

Step 8. If $c \in (-1, 1)$, compute $\mathrm{Ci}(|(1-c)\,\omega|)$, $\mathrm{Ci}(|(1+c)\,\omega|)$, $\mathrm{Si}((1-c)\,\omega)$ and $\mathrm{Si}((1+c)\,\omega)$ to obtain $⨍_{-1}^{1} e^{ic\omega}\,dt/(t-c)$ in (2.1). If $c = \pm 1$, compute $\mathrm{Ci}(2|\omega|)$ and $\mathrm{Si}(2\omega)$ to obtain $⨎_{-1}^{1} e^{i\omega t}/(t-c)\,dt$ in (2.3).

Step 9. Evaluate $p_n(c)$ by Clenshaw's algorithm‡.

Step 10. Evaluate $Q(p_n; c, \omega)$ by (1.6), where $I(p_n; c, \omega)$ is replaced by the approximation $I^{[N]}(p_n; c, \omega)$ in Step 7 when $n - 1 > |\omega|$.

† The *discrete cosine transform* (DCT type 1) with the FFT (cf. [19, p.229, p.232]) for a_k =`a(k+1)` ($0 \le k \le n$), Chebyshev coefficients of the polynomial p_n interpolating f, is written as a Matlab script,

```
t=linspace(0,pi,n+1);  t=cos(t);  y=f(t);
g=[y y(n:-1:2)];  g=fft(g')/n;  a=g(1:n+1);
```

‡ *Clenshaw's algorithm* (cf. [24, p.166],[3, p.27],[34, p.157]) for evaluating $p_n(c)$ at $t = c$ in (1.2) is as follows. Set $y_{n+1} = 0$ and $y_n = a_n/2$. Compute y_k by

$$y_k = 2\,c\,y_{k+1} - y_{k+2} + a_k \qquad (k = n-1, \cdots, 0). \tag{5.1}$$

Then, $p_n(c) = (y_0 - y_2)/2$.

6. Error analysis

We estimate the error of the approximation $Q(p_n; c, \omega)$ in (1.6) to the integrals $Q(f; c, \omega)$ in (1.1). For $f \in C^1[-1, 1]$, the uniform convergence of the method is proved in [13]. Here, assuming that f is analytic on a region in the complex plane, we show the uniform convergence using a tighter error bound. A similar error analysis is in [12], see also [11]. See [35] for uniform approximations to hypersingular finite-part integrals.

6.1. Uniform error bound

Let $\mathcal{E}_\rho$ denote an ellipse in the complex plane: $|z-1| + |z+1| = \rho + \rho^{-1}$, whose foci are at $z = \pm 1$ and the sum of semi-axes is $\rho > 1$. Let $u = \rho\, e^{i\theta}$ ($0 \le \theta \le 2\pi$), then we have

$$\mathcal{E}_\rho : z = \frac{1}{2}(u + u^{-1}). \tag{6.1}$$

Let $W_{n+1}(x)$ be a polynomial of degree $n+1$ defined by

$$W_{n+1}(t) = T_{n+1}(t) - T_{n-1}(t) = 2\,(t^2 - 1)\,U_{n-1}(t) \quad (n \ge 1). \tag{6.2}$$

Then, the Chebyshev points $\cos(\pi j/n)$ $(0 \le j \le n)$ are zeros of $W_{n+1}(x)$. Lemma 6.1 is given in [3, p.150].

Lemma 6.1 (cf. [3])**.** *Suppose that $f(z)$ is single-valued and analytic inside and on $\mathcal{E}_\rho$ defined by (6.1). Then, the error $e_n(t)$ of the interpolating polynomial $p_n(t)$ in (1.2) is expressed in terms of a contour integral,*

$$e_n(t) = f(t) - p_n(t) = \frac{W_{n+1}(t)}{2\pi i} \oint_{\mathcal{E}_\rho} \frac{f(z)}{W_{n+1}(z)\,(z-t)}\, dz. \tag{6.3}$$

Remark 6.2. *If $f(z)$ is analytic on $[-1,1]$, then there exists a sufficiently small $\rho > 1$ such that $f(z)$ is analytic inside and on the ellipse $\mathcal{E}_\rho$.*

Let E_1 and E_2 be defined by

$$E_1 = \int_{-1}^{1} \frac{e_n(t) - e_n(c)}{t-c} e^{i\omega t}\, dt, \tag{6.4}$$

$$E_2 = \begin{cases} e_n(c) \displaystyle ⨍_{-1}^{1} \frac{e^{i\omega t}}{t-c}\, dt, & c \in (-1,1), \\ 0, & c = \pm 1, \end{cases} \tag{6.5}$$

respectively. Then, the error of the approximation $Q(p_n; c, \omega)$ is given by

$$|Q(f; c, \omega) - Q(p_n; c, \omega)| = |Q(e_n; c, \omega)| = |E_1 + E_2| \le |E_1| + |E_2|. \tag{6.6}$$

Now, let us bound E_1 and E_2.

Lemma 6.3. *Suppose that $f(z)$ is single-valued and analytic inside and on $\mathcal{E}_\rho$ defined by (6.1) and let $M = \max_{z\in\mathcal{E}_\rho} |f(z)|$. Then, for $c \in [-1,1]$, we bound E_1 in (6.4) by*

$$|E_1| \le \frac{8M\rho}{\pi(\rho-1)^2\,(\rho^n - \rho^{-n})} \Big\{ 2(2n+1) + \pi \log\Big(\frac{\rho+1}{\rho-1}\Big) \Big\} \quad (c \in [-1,1]). \tag{6.7}$$

Proof. In view of Lemma 6.1 and (6.4) we obtain

$$E_1 = \frac{1}{2\pi i} \oint_{\mathcal{E}_\rho} \frac{f(z)}{W_{n+1}(z)} \int_{-1}^{1} \frac{1}{t-c} \Big\{ \frac{W_{n+1}(t)}{z-t} - \frac{W_{n+1}(c)}{z-c} \Big\} e^{i\omega t}\, dt\, dz. \tag{6.8}$$

Let G_1 be defined by

$$G_1 = \max_{\substack{z\in\mathcal{E}_\rho \\ c\in[-1,1]}} \int_{-1}^{1} \Big| \frac{1}{t-c} \Big\{ \frac{W_{n+1}(t)}{z-t} - \frac{W_{n+1}(c)}{z-c} \Big\} \Big|\, dt. \tag{6.9}$$

Then, from (6.8) we have

$$|E_1| \le \frac{MG_1}{2\pi} \oint_{\mathcal{E}_\rho} \Big| \frac{dz}{W_{n+1}(z)} \Big|. \tag{6.10}$$

We claim that

$$\oint_{\mathcal{E}_\rho} \Big|\frac{dz}{W_{n+1}(z)}\Big| \le \frac{2\pi}{\rho^n - \rho^{-n}}, \tag{6.11}$$

and

$$G_1 \le \frac{8\rho}{\pi(\rho-1)^2}\Big\{2(2n+1) + \pi\log\Big(\frac{\rho+1}{\rho-1}\Big)\Big\}. \tag{6.12}$$

Then, inserting (6.11) and (6.12) into (6.10) proves the lemma.

It remains to verify (6.11) and (6.12). Verification of (6.11) is simple. For

$$z = \frac{1}{2}(u+u^{-1}) \in \mathcal{E}_\rho, \quad u = \rho\, e^{i\theta},$$

(6.1), we have $T_n(z) = \frac{1}{2}(u^n + u^{-n})$ (cf. [3, p.14]). Consequently, from (6.2) we have

$$W_{n+1}(z) = \frac{1}{2}(u^{n+1}+u^{-n-1}) - \frac{1}{2}(u^{n-1}+u^{-n+1}) = \frac{1}{2}(u^n - u^{-n})(u-u^{-1}). \tag{6.13}$$

We verify (6.11) since in view of $dz = \frac{1}{2}(1-u^{-2})du$ and (6.13) we have

$$\oint_{\mathcal{E}_\rho} \Big|\frac{dz}{W_{n+1}(z)}\Big| = \oint_{|u|=\rho} \Big|\frac{du}{(u^n - u^{-n})u}\Big| \le \oint_{|u|=\rho} \frac{du}{(\rho^n-\rho^{-n})\rho} = \frac{2\pi}{\rho^n-\rho^{-n}}. \tag{6.14}$$

Verification of (6.12) requires some manipulations. Since

$$\frac{1}{t-c}\Big\{\frac{W_{n+1}(t)}{z-t} - \frac{W_{n+1}(c)}{z-c}\Big\} = \frac{W_{n+1}(t)-W_{n+1}(c)}{(z-t)(t-c)} + \frac{W_{n+1}(c)}{(z-t)(z-c)},$$

$$\min_{\substack{z\in\mathcal{E}_\rho\\ t\in[-1,1]}} |z-t| = \frac{\rho+\rho^{-1}}{2} - 1 = \frac{(\rho-1)^2}{2\rho}, \tag{6.15}$$

$$\max_{z\in\mathcal{E}_\rho} \int_{-1}^{1} \frac{dt}{|z-t|} = 2\log\Big(\frac{\rho+1}{\rho-1}\Big), \tag{6.16}$$

see (7.12) in [8] for (6.16), and $|W_{n+1}(c)| \le 2$ ($c \in [-1,1]$), from (6.9) we have

$$\begin{aligned} G_1 &\le \max_{\substack{z\in\mathcal{E}_\rho\\ c\in[-1,1]}} \int_{-1}^{1} \Big|\frac{W_{n+1}(t)-W_{n+1}(c)}{(z-t)(t-c)}\Big| dt + \max_{\substack{z\in\mathcal{E}_\rho\\ c\in[-1,1]}} \Big|\frac{W_{n+1}(c)}{z-c}\Big| \int_{-1}^{1} \frac{dt}{|z-t|} \\ &\le \frac{2\rho}{(\rho-1)^2}\Big\{\max_{c\in[-1,1]} \int_{-1}^{1} \Big|\frac{W_{n+1}(t)-W_{n+1}(c)}{t-c}\Big| dt + 4\log\Big(\frac{\rho+1}{\rho-1}\Big)\Big\}. \end{aligned} \tag{6.17}$$

Since

$$U_k(t) - U_{k-2}(t) = 2T_k(t) \quad (k \ge 2), \quad U_1(t) = 2T_1(t),$$

(cf. [34, p.9]), and

$$T_{n+1}(t) - T_{n+1}(c) = 2(t-c) \sum_{k=0}^{n}{}' U_{n-k}(t) T_k(c),$$

(cf. [36]), from (6.2) we have

$$\frac{W_{n+1}(t) - W_{n+1}(c)}{t-c} = 4 \sum_{j=0}^{n}{}'' T_{n-j}(c)\, T_j(t). \tag{6.18}$$

In view of (6.18) and the relations

$$\int_{-1}^{1} |T_0(t)|\, dt = 2 < \frac{8}{\pi}, \int_{-1}^{1} |T_1(t)| dt = 1 < \frac{4}{\pi}, \int_{-1}^{1} |T_k(t)|\, dt \le \frac{4}{\pi}\ (k \ge 2),$$

(cf. [34, p.38]), we have

$$\int_{-1}^{1} \Big| \frac{W_{n+1}(t) - W_{n+1}(c)}{t-c} \Big|\, dt \le 4 \sum_{j=0}^{n}{}'' \int_{-1}^{1} |T_j(t)|\, dt \le \frac{8(2n+1)}{\pi}. \tag{6.19}$$

Using (6.19) in (6.17) verifies (6.12). □

Lemma 6.4. *Under the same assumption as in Lemma 6.3, we bound* E_2 *in (6.5) by*

$$|E_2| \le \frac{16 M\rho}{(\rho-1)^2\,(\rho^n - \rho^{-n})} \quad (c \in [-1,1]). \tag{6.20}$$

Proof. When $c = \pm 1$, in view of $W_{n+1}(c) = 0$ we have $e_n(c) = 0$ in (6.3), consequently $E_2 = 0$ in (6.5). Now, we assume that $c \in (-1,1)$. From Lemma 2.1 and (6.5) we have

$$|E_2| \le |e_n(c)| \times \tag{6.21}$$
$$\sqrt{\{\mathrm{Ci}(|(1-c)\omega|) - \mathrm{Ci}(|(1+c)\omega|)\}^2 + \{\mathrm{Si}((1-c)\omega) + \mathrm{Si}((1+c)\omega)\}^2}.$$

In view of (6.3), (6.14) and (6.15) we have

$$\begin{aligned} |e_n(c)| &\le \frac{|W_{n+1}(c)|}{2\pi} \oint_{\mathcal{E}_\rho} \Big| \frac{f(z)\, dz}{W_{n+1}(z)(z-c)} \Big| \\ &\le \frac{M\rho\, |W_{n+1}(c)|}{\pi(\rho-1)^2} \oint_{\mathcal{E}_\rho} \Big| \frac{dz}{W_{n+1}(z)} \Big| \le \frac{2M\rho\, |W_{n+1}(c)|}{(\rho-1)^2(\rho^n - \rho^{-n})}. \end{aligned} \tag{6.22}$$

In view of Lemma 2.1 we see that

$$|\mathrm{Ci}(|(1-c)\omega|)-\mathrm{Ci}(|(1+c)\omega|)|=\Big|\int_{|(1+c)\omega|}^{|(1-c)\omega|}\frac{\cos t}{t}\,dt\Big|$$
$$\le\Big|\int_{|(1+c)\omega|}^{|(1-c)\omega|}\frac{1}{t}\,dt\Big|=\Big|\log\frac{1-c}{1+c}\Big|\qquad(c\in(-1,1)).\qquad(6.23)$$

In [32, p.244] it is shown that $0\le \mathrm{Si}(t)\le \mathrm{Si}(\pi)=1.8519370\cdots$. From (6.2) setting $c=\cos\vartheta$ we have

$$|W_{n+1}(c)|=2|\sin\vartheta\sin n\vartheta|\le 2|\sin\vartheta|=2\sqrt{1-c^2}.$$

Using these results and (6.22) and (6.23) in (6.21) we have

$$|E_2|\le\frac{4M\rho}{(\rho-1)^2(\rho^n-\rho^{-n})}\sqrt{\phi(c)},\qquad(6.24)$$
$$\phi(c)=(1-c^2)\Big\{\Big(\log\frac{1-c}{1+c}\Big)^2+4s^2\Big\},\qquad(6.25)$$

where we set $s=\mathrm{Si}(\pi)$. We claim that $\max_{-1<c<1}\phi(c)=4s^2<16$. Then, (6.20) follows from (6.24).

Verifying that $\max_{-1<c<1}\phi(c)=4s^2$ is simple. Since $\phi(c)$ is an even function, it is enough to examine only the case, $0\le c<1$. Since $\phi(0)=4s^2$, $\phi'(0)=0$, $\phi''(0)=8(1-s^2)<0$ and

$$\phi^{(3)}(c)=\frac{16\log\{(1-c)/(1+c)\}}{(1-c^2)^2}<0\quad(0<c<1),$$

we have $\max_{0\le c\le 1}\phi(c)=\phi(0)=4s^2$. □

In view of (6.6), Lemmas 6.3 and 6.4, Theorem 6.5 for the error bound of the approximation $Q(p_n;c,\omega)$ immediately follows.

Theorem 6.5. *Under the same assumption as in Lemma 6.3, the error of the approximations $Q(p_n;c,\omega)$ is bounded uniformly by*

$$|Q(f;c,\omega)-Q(p_n;c,\omega)|\le\frac{8M\rho\left[\,2(2n+1+\pi)+\pi\log\{(\rho+1)/(\rho-1)\}\,\right]}{\pi(\rho-1)^2\,(\rho^n-\rho^{-n})}$$
$$\sim(32M/\pi)\,n\rho^{-n-1}=O(n\rho^{-n})\qquad(n\to\infty).\quad(6.26)$$

While our error bound does not depend on ω, in the approximation method given in [17] that is of a different type from ours, the absolute error of the approximation decreases very fast as $|\omega|$ grows.

6.2. *An illustrative evidence of uniform error bound*

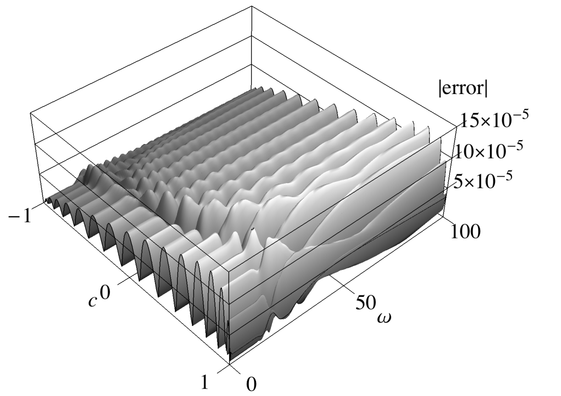

Figure 1: The error of the approximation at $n+1=17$ nodes for the integral $Q(f;c,\omega)$ as a function of c and ω, where $f(t)=(1-\alpha^2)/(1-2\alpha t+\alpha^2)$ and $\alpha=0.5$.

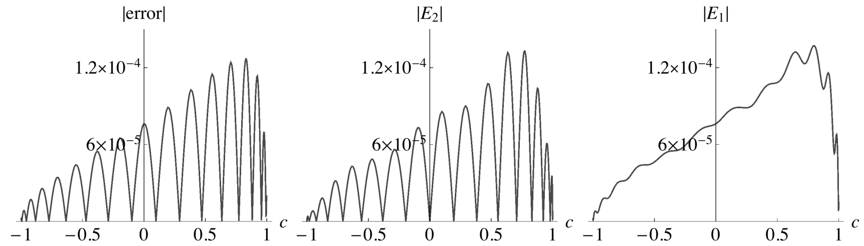

Figure 2: The error (*left*) of the approximation at $n+1=17$ nodes for the integral $Q(f;c,10)$ as a function of c, $|E_2|$ (6.5) (*center*) and $|E_1|$ (6.4) (*right*), where $f(t)=(1-\alpha^2)/(1-2\alpha t+\alpha^2)$ and $\alpha=0.5$.

Figure 1 illustrates the absolute error $|Q(f;c,\omega)-Q(p_n;c,\omega)|$ of the approximation $Q(p_n;c,\omega)$ at $n+1=17$ nodes to the integral $Q(f;c,\omega)$ for $f(t)=(1-\alpha^2)/(1-2\alpha t+\alpha^2)$ with $\alpha=0.5$, $c\in[-1,1]$ and $1\le\omega\le100$. Figure 2 shows the error $|Q(f;c,\omega)-Q(p_n;c,\omega)|=|Q(e_n;c,\omega)|$, $|E_2|$ and $|E_1|$ for $\omega=10$ and Figure 3 for $\omega=100$. (Recall that the quadrature error in magnitude is $|Q(e_n;c,\omega)|=|E_1+E_2|\le|E_1|+|E_2|$). We observe that the error is uniformly bounded, independently of the values of c and ω. Further,

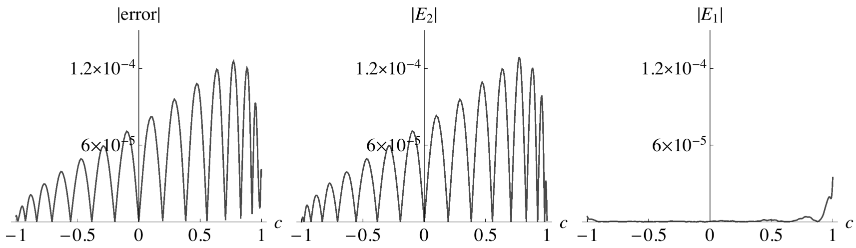

Figure 3: The error (*left*) of the approximation at $n+1 = 17$ nodes for the integral $Q(f; c, 100)$ as a function of c, $|E_2|$ (6.5) (*center*) and $|E_1|$ (6.4) (*right*), where $f(t) = (1-\alpha^2)/(1-2\alpha t+\alpha^2)$ and $\alpha = 0.5$.

$|E_2|$ vanishes when c coincides with the nodes, where $e_n(c) = 0$ in (6.5). From Figures 2 and 3 we observe that $|E_1|$ and $|E_2|$ are comparable for small $|\omega|$ while $|E_1|$ is much smaller than $|E_2|$ for larger $|\omega|$, indeed, $|E_1| = O(1/\omega)$ as $|\omega| \to \infty$. Consequently, $|E_2|$ dominantly contributes to the quadrature error (6.6), particularly for large $|\omega|$.

7. Numerical examples

In this section, by numerical examples we demonstrate the performance of our algorithm. The numerical computations are carried out with about 15 significant digits by using Mathematica (Ver. 11.1.1.0) on the iMac (3.2GHz, Intel Core i5). The values that we regard as exact are obtained with 100 significant digits computation on Mathematica.

We compute five types of oscillatory CPV integrals,

$$Q(f_m; c, \omega) = \fint_{-1}^{1} \frac{f_m(t)\, e^{i\omega t}}{t-c}\, dt, \quad m = 1, \ldots, 5,$$

for $c = 0.9$ and $\omega = 10$, 1000, where the functions $f_m(t)$ are given by

$$f_1(t) = e^{\alpha(t-1)}, \qquad \alpha = 4, 16, \tag{7.1}$$

$$f_2(t) = e^{i2\pi\alpha t}, \qquad \alpha = 8, 16, \tag{7.2}$$

$$f_3(t) = \frac{1-\alpha^2}{1-2\alpha t+\alpha^2}, \quad \alpha = 0.8, 0.9, \tag{7.3}$$

$$f_4(t) = \frac{1}{t^2+\alpha^2}, \qquad \alpha = 1/4, 1/8, \tag{7.4}$$

$$f_5(t) = (1-t^2)^{3/2}, \tag{7.5}$$

respectively. We compute oscillatory HFP integrals with $c = 1$ and $\omega = 10, 1000$ as well and omit the details since obtained results for relative errors are very similar to Figures 4∼8 below for the oscillatory CPV integrals. We note that the uniform dependence of the approximation $Q(p_n; c, \omega)$ on the value of n is common for most Clenshaw-Curtis like methods, including [12–14].

The function in (7.1) is an entire function, (7.2) an oscillatory function, (7.3) the generating function of the Chebyshev polynomial (cf. [22], [34, p.41]), (7.4) a peaked function and (7.5) a function having singularities of the second-derivative at both ends of the integration interval.

The integrals $Q(f_m; 0.9, \omega)$ $(m = 1, \ldots, 4)$ except for $Q(f_5; 0.9, \omega)$ with their exact values (for $\omega = 10, 1000$) are expressed by

$$Q(f_1; 0.9, 10) = \Phi_I(0.9, 10 - i\alpha)\, e^{-\alpha}$$
$$= \begin{cases} -1.1256339442498735738 - 1.2174807464660865793i & (\alpha = 4), \\ -0.79432599720832534426 - 0.27066674690448630758i & (\alpha = 16), \end{cases}$$

$$Q(f_1; 0.9, 1000) = \Phi_I(0.9, 1000 - i\alpha)\, e^{-\alpha}$$
$$= \begin{cases} -2.0930127016937171914 + 0.13383440410443484 88i & (\alpha = 4), \\ -0.62458187202084588291 + 0.036445320538909205 37i & (\alpha = 16), \end{cases}$$

$$Q(f_2; 0.9, 10) = \Phi_R(0.9, 10 + 2\pi\alpha)$$
$$= \begin{cases} 2.2610032378003899837 - 1.9621437455729568795i & (\alpha = 8), \\ 2.6905839749162102713 + 1.6377598933795257917i & (\alpha = 16), \end{cases}$$

$$Q(f_2; 0.9, 1000) = \Phi_R(0.9, 1000 + 2\pi\alpha)$$
$$= \begin{cases} -1.1592005127547868638 - 2.9226666731058770971i & (\alpha = 8), \\ 2.4207591820350711974 - 2.0163447425643565369i & (\alpha = 16), \end{cases}$$

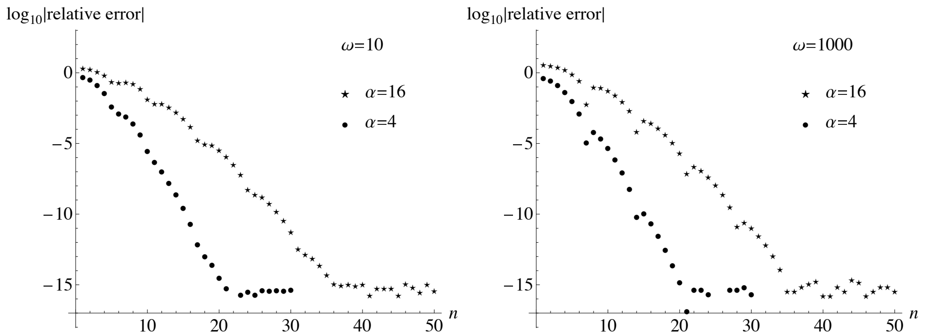

Figure 4: The relative errors of the approximations to the integrals $Q(f_1; 0.9, \omega)$ for $f_1(t) = e^{\alpha(t-1)}$ with $\alpha = 4, 16$ and for $\omega = 10$ (*left*) and 1000 (*right*).

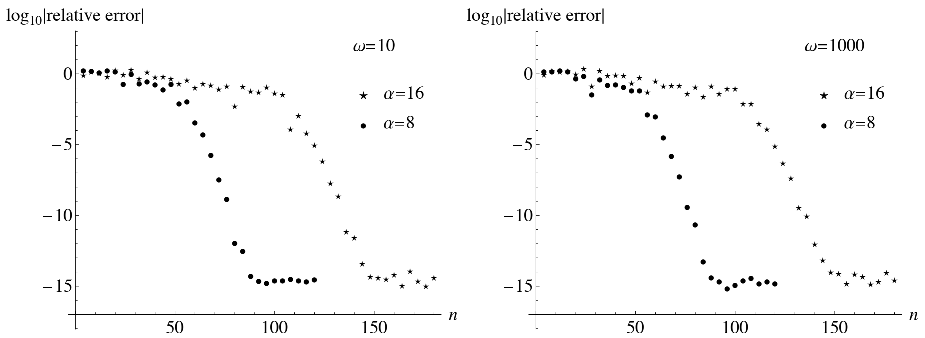

Figure 5: The relative errors of the approximations to the integrals $Q(f_2; 0.9, \omega)$ for $f_2(t) = e^{i2\pi\alpha t}$ with $\alpha = 8, 16$ and for $\omega = 10$ (*left*) and 1000 (*right*).

$$Q(f_3; 0.9, 10) = \frac{1-\alpha^2}{2\alpha(r-c)}\{\Phi_C(r, 10) - \Phi_R(0.9, 10)\}$$
$$= \begin{cases} -5.1726063581663838102 - 3.3490166983645622408i & (\alpha = 0.8), \\ -4.1057284162415934661 - 2.2927376525987158576i & (\alpha = 0.9), \end{cases}$$
$$r := (\alpha + 1/\alpha)/2,$$

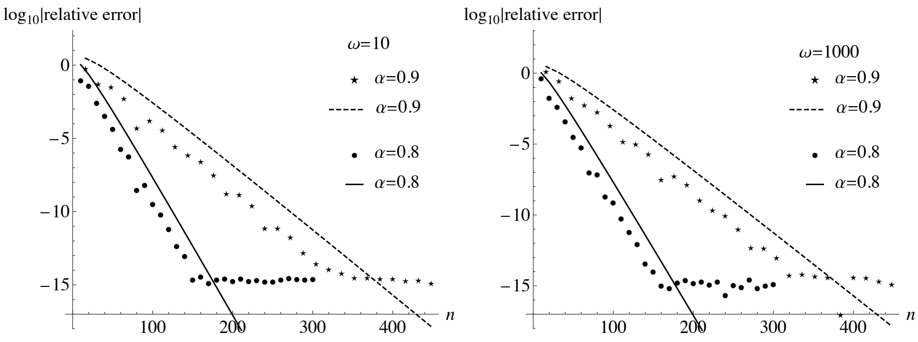

Figure 6: The relative errors of the approximations to the integrals $Q(f_3; 0.9, \omega)$ for $f_3(t) = (1-\alpha^2)/(1-2\alpha t+\alpha^2)$ with $\alpha = 0.8, 0.9$ and for $\omega = 10$ (*left*) and 1000 (*right*). The solid and broken curves represent $n\rho^{-n}$ for the parameter ρ of the ellipse $\mathcal{E}_\rho$ (6.1).

$$
\begin{aligned}
Q(f_3; 0.9, 1000) &= \frac{1-\alpha^2}{2\alpha(r-c)}\{\Phi_C(r, 1000) - \Phi_R(0.9, 1000)\} \\
&= \begin{cases} -5.5667603364747801020 + 0.3263089276087251178i & (\alpha = 0.8), \\ -2.9692938030411933993 + 0.1295150056147923552i & (\alpha = 0.9), \end{cases} \\
r &:= (\alpha + 1/\alpha)/2,
\end{aligned}
$$

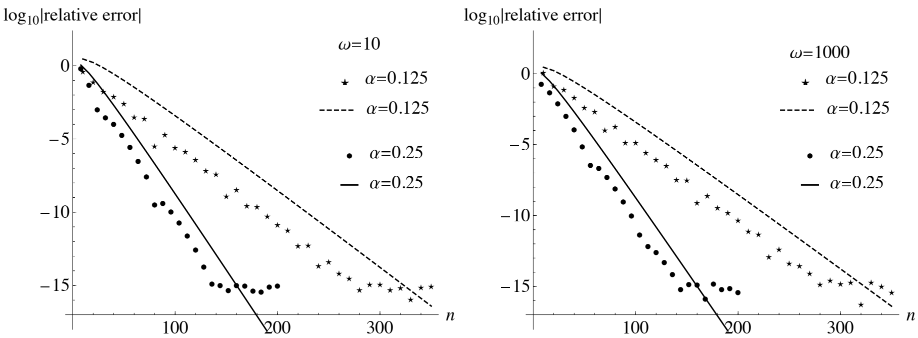

Figure 7: The relative errors of the approximations to the integrals $Q(f_4; 0.9, \omega)$ for $f_4(t) = 1/(t^2+\alpha^2)$ with $\alpha = 1/4, 1/8$ and for $\omega = 10$ (*left*) and 1000 (*right*). The solid and broken curves represent $n\rho^{-n}$ for the parameter ρ of the ellipse $\mathcal{E}_\rho$ (6.1).

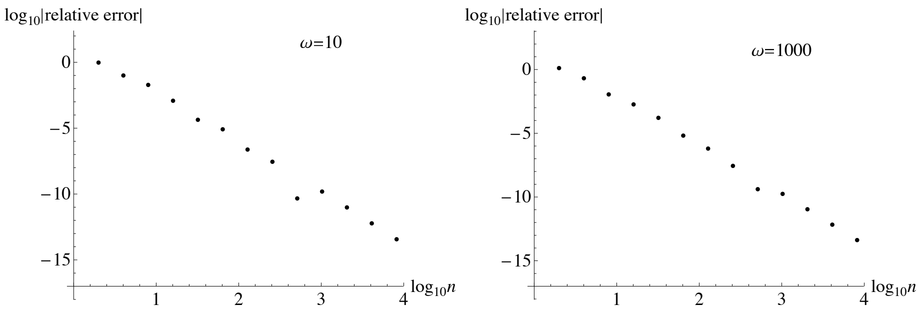

Figure 8: The relative errors of the approximations to the integrals $Q(f_5; 0.9, \omega)$ for $f_5(t) = (1-t^2)^{3/2}$ and for $\omega = 10$ (*left*) and 1000 (*right*).

$$Q(f_4; 0.9, 10) = \frac{1}{2i\alpha}\left\{\frac{\Phi_C(i\alpha, 10) - \Phi_R(0.9, 10)}{i\alpha - 0.9} + \frac{\Phi_C(-i\alpha, 10) - \Phi_R(0.9, 10)}{i\alpha + 0.9}\right\}$$
$$= \begin{cases} -2.5024802215496231205 - 2.9010231004723840715i & (\alpha = 1/4), \\ -9.3674757276956166055 - 3.8530400190128827327i & (\alpha = 1/8), \end{cases}$$

$$Q(f_4; 0.9, 1000) = \frac{1}{2i\alpha}\left\{\frac{\Phi_C(i\alpha, 1000) - \Phi_R(0.9, 1000)}{i\alpha - 0.9} + \frac{\Phi_C(-i\alpha, 1000) - \Phi_R(0.9, 1000)}{i\alpha + 0.9}\right\}$$
$$= \begin{cases} -3.5854608269985750047 + 0.2328694521987087191i & (\alpha = 1/4), \\ -3.7891040784328703465 + 0.2461500798066019073i & (\alpha = 1/8), \end{cases}$$

where $\Phi_R(c,\omega)$, $\Phi_I(c,\omega)$ and $\Phi_C(c,\omega)$ are defined by

$$\Phi_R(c,\omega) = ⨍_{-1}^{1} \frac{e^{i\omega t}}{t-c}\,dt, \quad c \in (-1,1),\ \omega \in \mathbb{R} \setminus \{0\},$$
$$\Phi_I(c,\omega) = ⨍_{-1}^{1} \frac{e^{i\omega t}}{t-c}\,dt, \quad c \in (-1,1),\ \omega \in \mathbb{C} \setminus \mathbb{R},$$
$$\Phi_C(c,\omega) = \int_{-1}^{1} \frac{e^{i\omega t}}{t-c}\,dt, \quad c \in \mathbb{C} \setminus (-1,1),\ \omega \in \mathbb{R},$$

respectively and are written by

$$\Phi_R(c,\omega) = e^{i\omega c}\big[\mathrm{Ci}(|(1-c)\omega|) - \mathrm{Ci}(|(1+c)\omega|) + iS\big]$$
$$\Phi_I(c,\omega) = e^{i\omega c}\big[\mathrm{Ci}((1-c)\omega) - \mathrm{Ci}((1+c)\omega) + iS\big],$$
$$\Phi_C(c,\omega) = e^{i\omega c}\big[\mathrm{Ci}((1-c)\omega) - \mathrm{Ci}(-(1+c)\omega) + iS\big],$$

respectively. Here we set $S = \mathrm{Si}((1-c)\omega)+\mathrm{Si}((1+c)\omega)$. On the other hand, the exact value for the integral $Q(f_5; c, \omega)$ is obtained as follows. First, we rewrite $Q(f_5; c, \omega)$ as

$$Q(f_5; c, \omega) = \int_{-1}^{1} \frac{f_5(t) - f_5(c)}{t-c} e^{i\omega t}\, dt + f_5(c)\Phi_R(c, \omega),$$

and compute the integral on the right-hand side of the equation above by using the routine Nintegrate in Mathematica with the required tolerance 10^{-30}. Then, we have, for $c = 0.9$ and $\omega = 10, 1000$,

$$Q(f_5; 0.9, 10) = -0.08561094788020693315 - 0.28387269290696526616i$$
$$Q(f_5; 0.9, 1000) = -0.25961337137912856956 + 0.01723611261373691240i$$

From Figure 8, we observe that the error of the approximation to $Q(f_5; c, \omega)$ of $f_5(t)$ with singularities of the second-derivative decreases slowly like $O(1/n^{\alpha})$, $\alpha \approx 4$ as n grows.

8. Concluding remarks

We presented an efficient quadrature rule of Clenshaw-Curtis type for approximating Cauchy principal value integrals of oscillatory functions and oscillatory Hadamard finite-part integrals. An algorithm was provided in a form easy to implement. We incorporated an improved version of the routine for the oscillatory integrals [8] into the algorithm. We proved that the error of the approximation is bounded independently of the values of $c \in [-1, 1]$ and $\omega \neq 0$. Numerical examples illustrated the efficiency of our method. The present method has an application to the approximations of oscillatory Hilbert transforms on the interval $[0, \infty)$ and $(-\infty, \infty)$ (cf. [14]).

Appendix A. Proof of Lemma 1.1

We outline the proof of Lemma 1.1 that is almost the same as that in our previous method [8] with the normalization relation $\varphi(-1) = 0$. For $\varphi(x)$ in

(1.11) and $\varphi^{[N]}(x)$ in (1.15) let $\varepsilon^{[N]}(x) = \varphi^{[N]}(x) - \varphi(x)$. From the proof of Lemma 3.1 in [8] we derive

$$\varepsilon^{[N]}(1)e^{i\omega} - \varepsilon^{[N]}(-1)e^{-i\omega} = \frac{i\omega}{2} d_N^{[N]} \int_{-1}^{1} U_N(x) e^{i\omega x}\, dx. \tag{A.1}$$

Since in view of (1.12) and (1.16) we have

$$I(p_n; c, \omega) = \frac{1}{i\omega}\{\varphi(1)e^{i\omega} - \varphi(-1)e^{-i\omega}\},$$
$$I^{[N]}(p_n; c, \omega) = \frac{1}{i\omega}\{\varphi^{[N]}(1)e^{i\omega} - \varphi^{[N]}(-1)e^{-i\omega}\},$$

from (A.1) it follows that

$$I(p_n; c, \omega) - I^{[N]}(p_n; c, \omega) = \frac{-1}{2} d_n^{[N]} \int_{-1}^{1} U_N(x) e^{i\omega x}\, dx.$$

So, we verify Lemma 1.1 since $|\int_{-1}^{1} U_N(x) e^{i\omega x}\, dx| \leq 2$, see [8]. $\square$

Appendix B. Proving that $J_M(\omega) > 0$

For a given $|x| > 0$, let $n = \lfloor |x| \rfloor$. Then, for the Bessel function $J_n(x)$ of order n, we prove that $|J_n(x)| > 0$ for $0 < |x| \leq n+1$. Since $J_n(-x) = (-1)^n J_n(x)$ (see (B.2) below), it suffices to show that $J_n(x) > 0$ for $0 < x \leq n+1$. Since $J_0(x) > 0$ $(0 < x \leq 1)$ (cf. [32, p.359]), we prove the case $n \geq 1$. We claim that for an arbitrary integer $\nu \geq 1$,

$$J_\nu(x) > 0, \quad J_\nu'(x) > 0 \quad (0 < x \leq \nu), \tag{B.1}$$

where $J_\nu'(x)$ is the derivative of $J_\nu(x)$. Then, since $J_n(x) = J_{n+2}(x) + 2J_{n+1}'(x)$ (cf. [32, 9.1.27]), we verify that $J_n(x) > 0$ for $0 < x \leq n+1$.

It remains to verify (B.1). Denote by $j_{\nu,1}$ the smallest positive zero of $J_\nu(x)$ and by $j_{\nu,1}'$ that of $J_\nu'(x)$. Then, $\nu \leq j_{\nu,1}' < j_{\nu,1}$ (cf. [32, 9.5.2]). Since (cf. [32, 9.1.10])

$$J_\nu(x) = \left(\frac{x}{2}\right)^\nu \sum_{k=0}^{\infty} \frac{(-x^2/2)^k}{k!, \Gamma(\nu+k+1)} = \frac{1}{\Gamma(\nu+1)}\left(\frac{x}{2}\right)^\nu + O(x^{\nu+2}), \tag{B.2}$$

for sufficiently small $\epsilon > 0$, we have $J_\nu(\epsilon) > 0$, consequently $J_\nu(x) > 0$ for $0 < x < j_{\nu,1}$. In view of the fact that $\nu < j_{\nu,1}$, this verify the first relation of (B.1). Since from (B.2) we have $J_\nu'(x) = (\nu/2)(x/2)^{\nu-1}/\Gamma(\nu+1) + O(x^{\nu+1})$, we similarly verify the second relation of (B.1). $\square$

Acknowledgements

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References

[1] P. J. Davis, P. Rabinowitz, Methods of Numerical Integration, 2nd Edition, Academic Press, Orland, FL, 1984.

[2] C. W. Clenshaw, A. R. Curtis, A method for numerical integration on an automatic computer, Numer. Math. 2 (1960) 197–205.

[3] J. C. Mason, D. C. Handscomb, Chebyshev Polynomials, Chapman & Hall/CRC, Boca Raton, FL, 2003.

[4] T. Hasegawa, T. Torii, An automatic quadrature for Cauchy principal value integrals, Math. Comput. 56 (1991) 741–754.

[5] T. Hasegawa, T. Torii, Hilbert and Hadamard transforms by generalized Chebyshev expansion, J. Comput. Appl. Math. 51 (1994) 71–83.

[6] T. Hasegawa, Uniform approximations to finite Hilbert transform and its derivative, J. Comput. Appl. Math. 163 (2004) 127–138.

[7] P. Keller, Indefinite integration of oscillatory functions, Applic. Math. 25 (1999) 301–311.

[8] T. Hasegawa, H. Sugiura, A user-friendly method for computing indefinite integrals of oscillatory functions, J. Comput. Appl. Math. 315 (2017) 126–141.

[9] T. Hasegawa, T. Torii, An algorithm for nondominant solutions of linear second-order inhomogeneous difference equations, Math. Comput. 64 (1995) 1199–1214.

[10] T. Hasegawa, A. Sidi, An automatic integration procedure for infinite range integrals involving oscillatory kernels, Numer. Algor. 13 (1996) 1–19.

[11] T. Hasegawa, H. Sugiura, Uniform approximation to Cauchy principal value integrals with logarithmic singularity, J. Comput. Appl. Math. 327 (2018) 1–11.

[12] H. Wang, S. Xiang, Uniform approximation to Cauchy principal value integrals of oscillatory functions, Appl. Math. Comput. 215 (2009) 1886–1894.

[13] P. Keller, A practical algorithm for computing Cauchy principal values integrals of oscillatory functions, Appl. Math. Comput. 218 (2012) 4988–5001.

[14] H. Wang, L. Zhang, D. Huybrechs, Asymptotic expansions and fast computation of oscillatory Hilbert transforms, Numer. Math. 123 (2013) 709–743.

[15] G. E. Okecha, Quadrature formulae for Cauchy principal value integrals of oscillatory kind, Math. Comput. 49 (1987) 259–268.

[16] M. R. Capobianco, G. Criscuolo, On quadrature for Cauchy principal value integrals of oscillatory functions, J. Comput. Appl. Math. 156 (2003) 471–486.

[17] H. Wang, S. Xiang, On the evaluation of Cauchy principal value integrals of oscillatory functions, J. Comput. Appl. Math. 234 (2010) 95–100.

[18] W. M. Gentleman, Implementing Clenshaw-Curtis quadrature II: Computing the cosine transformation, Comm. ACM 15 (1972) 343–346.

[19] C. Van Loan, Computational Frameworks for the Fast Fourier Transform, Vol. 10 of Frontiers in Applied Mathematics, SIAM, Philadelphia, PA, 1992.

[20] J. A. C. Weideman, L. N. Trefethen, The kink phenomenon in Fejér and Clenshaw-Curtis quadrature, Numer. Math. 107 (2007) 707–727.

[21] L. N. Trefethen, Is Gauss quadrature better than Clenshaw-Curtis?, SIAM Rev. 50 (2008) 67–87.

[22] T. Hasegawa, H. Sugiura, Error estimate for a corrected Clenshaw-Curtis quadrature rule, Numer. Math. 130 (2015) 135–149.

[23] A. R. Krommer, C. W. Ueberhuber, Numerical Integration on Advanced Computer Systems, Vol. 848 of Lecture Notes in Computer Science edited by G. Goos and J. Hartmanis, Springer, Berlin Heidelberg, 1994.

[24] J. Wimp, Computation with Recurrence Relations, Pitman, Boston, MA, 1984.

[25] D. W. Lozier, Numerical solution of linear difference equations, Report NBSIR 80-1976, Math. Analysis Division, NBS, Washington DC, 1980.

[26] F. W. J. Olver, D. J. Sookne, Notes on backward recurrence algorithms, Math. Comput. 26 (1972) 941–947.

[27] J. Oliver, Relative error propagation in the recursive solution of linear recurrence relations, Numer. Math. 9 (1967) 323–340.

[28] F. W. J. Olver, Numerical solution of second-order linear difference equations, J. Res. NBS 71B (1967) 111–129.

[29] V. Domínguez, I. G. Graham, V. P. Smyshlyaev, Stability and error estimates of Filon-Clenshaw-Curtis rules for highly oscillatory integrals, IMA J. Numer. Anal. 31 (2011) 1253–1280.

[30] N. J. Higham, Accuracy and Stability of Numerical Algorithms, 2nd Edition, SIAM, Philadelphia, PA, 2002.

[31] W. H. Press, B. P. Flannery, S. A. Teukolsky, W. T. Vetterling, Numerical Recipes in C: The Art of Scientific Computing, Cambridge University Press, Cambridge, 1988.

[32] M. Abramowitz, I. A. Stegun (Eds.), Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables, Dover, New York, NY, 1965.

[33] N. M. Temme, Special Functions: An Introduction to the Classical Functions of Mathematical Physics, Wiley, New York, NY, 1996.

[34] T. J. Rivlin, Chebyshev Polynomials: From Approximation Theory to Algebra and Number Theory, 2nd Edition, Wiley, New York, NY, 1990.

[35] S. Xiang, C. Fang, Z. Xu, On uniform approximations to hypersingular finite-part integrals, J. Math. Anal. Appl. 435 (2016) 1210–1228.

[36] D. Elliott, Truncation errors in two Chebyshev series approximations, Math. Comput. 19 (1965) 234–248.